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# THE QUARTERLY JOURNAL OF MATHEMATICS

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# ON A SUM ANALOGOUS TO A GAUSS'S SUM

*By* L. J. MORDELL (*Manchester*)

[Received 7 March 1932]

**1.** LET $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ be any integers and put

$$f(x) = a_1 x^n + a_2 x^{n-1} + \ldots + a_n x. \tag{1}$$

Let $p$ be a prime, and write

$$S = \sum_{x=0}^{p-1} e^{2\pi i f(x)/p}, \tag{2}$$

so that no generality is lost by assuming that

$$a_1 \not\equiv 0 \pmod{p}, \qquad n < p-1.$$

All congruences in this paper are taken mod $p$ unless otherwise stated. I prove here that for large $p$,

$$S = O\left(p^{1-\frac{1}{n}}\right), \tag{3}$$

where the constant implied in $O$ depends only on $n$ and is independent of the $a$'s. In fact,

$$|S|^{2n} \leqslant \frac{n^n [n, p-1] p^{2n}}{p(p-1)} \qquad (p > n), \tag{4}$$

$$\leqslant \frac{(2n)!\,[n, p-1] p^{2n}}{n!\,p(p-1)} \qquad (p \leqslant n), \tag{4'}$$

where $[n, p-1]$ denotes the highest common factor of $n$ and $p-1$. This result is an improvement on the known one* found by Weyl's method

$$S = O\left(p^{1-\frac{1}{2^{n-1}}+\epsilon}\right),$$

where $\epsilon$ $(> 0)$ is arbitrarily small.

Next, let $l_1, l_2, \ldots, l_n$ be any integers and put

$$\phi(x) = a_1 x^{l_1} + a_2 x^{l_2} + \ldots + a_n x^{l_n}. \tag{5}$$

Write

$$S' = \sum_{x=1}^{p-1} e^{2\pi i \phi(x)/p}, \tag{6}$$

where if any $l$ is negative, $x^l$ is interpreted mod $p$, that is, $x^l = I$ where $Ix^{-l} \equiv 1 \pmod{p}$. It is obvious from (6) that no generality is lost in assuming that none of the $l$'s are congruent to zero mod $p-1$, that no two of the $l$'s are congruent to each other mod $p-1$, that

* Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 257, Satz 269.

$|l| < p-1$, that $n < p-1$, and that none of the $a$'s are congruent to zero mod $p$. Then I prove that

$$S' = O\left(p^{1-\frac{1}{2n}}\right). \tag{7}$$

where the constant implied in $O$ is independent of the $a$'s but depends on $n$ and the $l$'s. In fact,

$$|S'|^{2n} \leqslant \frac{(2n)!\, l_1 l_2 \ldots l_n [p-1, l_1, l_2, \ldots, l_n]}{n!n!} \frac{p^{2n}}{p-1}, \tag{8}$$

if all the $l$'s are positive, while if any $l$ is negative, the corresponding $l$ in the product $l_1 l_2 \ldots l_n$ must be replaced by $n|l|$.

The result when $n = 1$, with a better constant than in (8), is also* given in Landau. A slight modification of the case $n = 2$, $l_2 = 1$, namely, with $e^{2\pi i a_2 x/p}$ replaced by $\cos 2\pi a_2 x/p$, was discovered† by Hardy and Littlewood and published without proof. When $n = 2$, $l_1 = 1$, $l_2 = -1$, $S'$ becomes a Kloosterman sum, a subject which Estermann and Salié‡ have discussed in recent papers. My method of proof is similar to theirs, so that a part of the fundamental idea employed in this paper is not entirely new. But I discovered it in an application to some new and interesting theorems§ which obviously suggest new problems for investigation. Thus consider

$$y^m \equiv b_1 x^n + b_2 x^{n-1} + \ldots + b_{n+1} \pmod{p}$$

or, say,

$$y^m \equiv F(x),$$

where the $b$'s, $n$, $m$ are integers. Then the number $N$ of solutions of this congruence in $x$, $y$ when $p$ is large, is given by

$$N = p + O(p^{\phi(m,n)}),$$

where the constant implied in $O$ is independent of the $b$'s, and $\phi(m, n)$ depends only on $m$, $n$; e.g. $\phi(m, n) = 1$ is trivial.

Also

$$\phi(2, 3) = \phi(2, 4) = \tfrac{2}{3},$$

provided that $F(x)$ is not congruent to an algebraic square, that is, we exclude the case

$$F(x) \equiv c(c_1 x^2 + c_2 x + c_3)^2,$$

where the $c$'s are integers. A Riemann hypothesis for a zeta-function considered by Artin means that $\phi(2, 3) = \frac{1}{2}$.

* Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), p. 297, Satz 311.

† 'A new solution of Waring's problem', *Quart. J.*, 48 (1920), 277, footnote.

‡ *Math. Zeitschrift*, 34 (1931), 91–109, which also contains references.

§ The proofs will appear, I hope, in the *Math. Zeitschrift*.

Next, $\phi(3,3) = \frac{1}{2}$,

if $F(x)$ is not congruent to an algebraic cube.

Also, $\phi(m,3) = \frac{3}{4} \qquad (m > 3)$.

Further, $\phi(2,5) = \phi(2,6) = \frac{7}{8}, \qquad \phi(4,4) = \frac{5}{6}$,

if $F(x)$ is not congruent to an algebraic square.

**2.** I commence with the proof of (3). On changing the sign of $i$ in $S$ and multiplying,

$$|S|^2 = \sum_{x,y=0}^{p-1} e^{2\pi i\{f(x)-f(y)\}/p},$$

and so

$$|S|^{2n} = \sum_{x_r,y_r=0}^{p-1} e^{2\pi i f/p}, \tag{9}$$

say, where

$$f = \sum_{r=1}^{n} f(x_r) - \sum_{r=1}^{n} f(y_r),$$

and the summation in $x_r$, $y_r$ denotes that all the $x$'s and $y$'s run through the values 0, 1, 2,..., $p-1$. Clearly

$$f = \sum_{s=1}^{n} \Big[ a_s \sum_{r=1}^{n} (x_r^{n+1-s} - y_r^{n+1-s}) \Big], \tag{10}$$

Now consider the $a$'s in (9) as variables, and sum for each of them from 0 to $p-1$. The sum is zero, unless each of the coefficients of the $a$ in (10) is congruent to zero, when it becomes $p^n$. Hence

$$\sum_{a} |S|^{2n} = Np^n, \tag{11}$$

where $N$ is the number of solutions in the $2n$ variables $x_r$, $y_r$ of the $n$ congruences

$$\sum_{r=1}^{n} x_r^s \equiv \sum_{r=1}^{n} y_r^s \qquad (s = 1, 2,..., n). \tag{12}$$

From (12), it is easily deduced that, for $p > n$, the elementary symmetric functions of the $x$'s are congruent to the corresponding symmetric functions of the $y$'s. For write

$$s_1 = -\sum_{1}^{n} x_r, \qquad s_2 = \sum x_1 x_2, \qquad s_3 = -\sum x_1 x_2 x_3, \text{ etc.}$$

By Newton's formula, for $0 < m \leqslant n$,

$$\sum_{r=1}^{m} x_r^m + s_1 \sum_{r=1}^{n} x_r^{m-1} + ... + s_{m-1} \sum_{r=1}^{n} x_r + m s_m = 0.$$

Then, on taking $m = 1, 2,...$ in succession, the congruences (12) give

$$\left.\begin{aligned} \sum x_1 &\equiv \sum y_1, \\ 2\sum x_1 x_2 &\equiv 2\sum y_1 y_2, \\ 3\sum x_1 x_2 x_3 &\equiv 3\sum y_1 y_2 y_3, \end{aligned}\right\} \tag{13}$$

etc., and the required result since $p > n$. It follows now that each of the $x$'s is congruent to one of the $y$'s. For the $x$'s are the roots of the congruence in $x$,

$$(x-x_1)(x-x_2)\ldots(x-x_n) \equiv 0,$$

that is, $$x^n - x^{n-1}\sum x_1 + x^{n-2}\sum x_1 x_2 + \ldots \equiv 0,$$

or, from (13), $$x^n - x^{n-1}\sum y_1 + x^{n-2}\sum y_1 y_2 + \ldots \equiv 0,$$

or $$(x-y_1)(x-y_2)\ldots(x-y_n) \equiv 0.$$

Hence $x \equiv y_\lambda$, say. But, if the $y$'s in (12) assume arbitrary values, there are $p^n$ sets of values for the $y$'s and, for each set of $y$'s, there are at most $n^n$ sets of values for the $x$'s; and so $N \leqslant n^n p^n$.

Next, the same $|S|$ arises in (2) from $f(x)$ and also from the $p(p-1)$ functions $f(bx+c)-f(c)$, where $b$ takes the values 1, 2,..., $p-1$, and $c$ the values 0, 1,..., $p-1$. For with any of these values for $b$, $c$, $bx+c$ runs through a complete set of residues when $x$ does: that is

$$\sum_{x=0}^{p-1} e^{2\pi i f(x)/p} = \sum_{x=0}^{p-1} e^{2\pi i f(bx+c)/p}.$$

Some of the $p(p-1)$ functions $f(bx+c)-f(c)$ may be identically congruent to each other in $x$, but the same function cannot arise more than $[n, p-1]$ times, that is, for given $b'$, $c'$ there are not more than $[n, p-1]$ sets of values for $b$, $c$ such that

$$f(bx+c)-f(c) \equiv f(b'x+c')-f(c')$$

identically in $x$. On putting $X \equiv b'x+c'$, it suffices to show that there are at most $[n, p-1]$ sets of values for $B$, $C$ such that

$$f(BX+C)-f(X)$$

is identically independent of $X \bmod p$. The coefficient of $X^n$ gives $B^n \equiv 1$, that is, $[n, p-1]$ values for $B$, while the coefficient of $X^{n-1}$ shows that there is at most one value for $C$. Hence, for a given $a_1 \not\equiv 0$, $a_2$,..., $a_n$, we have from (11)

$$\frac{p(p-1)}{[n, p-1]}|S|^{2n} \leqslant n^n p^{2n},$$

which gives (3) and (4).

**3.** I now prove the results (7), (8) for $S'$ and (4') for $S$. The difference in the order of magnitude between $S$ and $S'$ is due to the fact that the same $|S'|$ arises from only the $p-1$ functions $\phi(bx)$, $(b = 1, 2,\ldots, p-1)$. These values of $b$ give $(p-1)/[p-1, l_1, l_2,\ldots, l_n]$ different functions, where $[p-1, l_1, l_2,\ldots, l_n]$, the greatest common

factor of $p-1, l_1, l_2, \ldots, l_n$, is the number of common solutions of the $n$ congruences $b^{l_1} \equiv 1, \quad b^{l_2} \equiv 1, \quad \ldots, \quad b^{l_n} \equiv 1.$

We have instead of (11)

$$\sum_a |S'|^{2n} = N' p^n, \tag{14}$$

where $N'$ is the number of solutions in the $2n$ variables $x_r$, $y_r$ of the $n$ congruences

$$\sum_{r=1}^{n} x_r^{l_s} \equiv \sum_{r=1}^{n} y_r^{l_s} \qquad (s = 1, 2, \ldots, n). \tag{15}$$

I prove that

$$N' \leqslant \frac{(2n)!}{n!n!} l_1 l_2 \ldots l_n p^n, \tag{16}$$

when all the $l$'s are positive, while, when any $l$ is negative, the result is given on replacing the corresponding $l$ in (16) by $n|l|$. We have then from (14) for a given set $a_1, a_2, \ldots, a_n$,

$$\frac{p-1}{[p-1, l_1, l_2, \ldots, l_n]} |S'|^{2n} \leqslant \frac{(2n)!}{n!n!} l_1 l_2 \ldots l_n p^{2n}.$$

This proves (7), (8); and (16) gives the constant in (4'), since $f(x)$ in (1) is the particular case of $\phi(x)$ in (5) when $l_1 = n$, $l_2 = n-1, \ldots,$ $l_n = 1$.

To prove (16), denote the right-hand side of (15) by $L_s$. The degree of this congruence in the $x$'s is $l_s$ if $l_s > 0$, and $n|l_s|$ if $l_s < 0$, as is obvious on replacing $x^{l_s}$ by $1/x^{-l_s}$. For arbitrary values of the $L$'s, the number of sets of solutions of the $x$'s does not exceed the product of the degrees of the congruences, a general result which we take for granted from the theory of elimination. Hence the general solution of (15) in the $2n$ variables $x_r$, $y_r$ apparently involves $n$ arbitrary parameters (by this we do not imply that integer values of the parameters give integer values for $x_r$, $y_r$).

This is in fact the case, for suppose that (15) has a solution with $n+1$ independent parameters. If the solution really depends upon $n+1$ parameters, there must be $n+1$ of the variables $x_r$, $y_r$ not connected by any polynomial congruence with coefficients depending only on $l_1, l_2, \ldots, l_n$, and actually involving the $n+1$ variables. There is no loss of generality in taking them to be $x_n$, $y_1$, $y_2, \ldots, y_n$ as the work is the same except for some signs which will not affect the result (16). This means that if we solve the first $n-1$ congruences of (15) for $x_1, x_2, \ldots, x_{n-1}$ in terms of $x_n, y_1, y_2, \ldots, y_n$, one of the sets of values for $x_1, x_2, \ldots, x_{n-1}$ must satisfy the $n$th congruence of (15) when

$x_n$, $y_1$, $y_2$,..., $y_n$ are considered as parameters. Hence a functional relation must connect the $n$ quantities

$$M_s = \sum_{r=1}^{n} y_r^{l_s} - x_n^{l_s} \qquad (s = 1, 2, ..., n)$$

of the form $$\sum A_{\lambda_1 \lambda_2 ...} M_1^{\lambda_1} M_2^{\lambda_2} ... \equiv 0, \tag{17}$$

or, say, $F \equiv 0$, where the $A$'s are integers independent of the $x$'s and $y$'s, are not all congruent to zero, and depend only on $l_1, l_2, ..., l_n$. We may suppose that $F$ is irreducible mod $p$, that is, there do not exist polynomials $F_1$, $F_2$, $F_3$ of the same type as $F$ such that

$$F = F_1 F_2 + p F_3$$

identically. For then $F \equiv 0$ could be replaced by $F_1 \equiv 0$ or $F_2 \equiv 0$.

But (17) cannot hold identically in the $y$'s and $x_{n+1}$. For if every negative exponent $l$ is replaced by $p-1+l$, which does not alter $y^l \bmod p$, (17) becomes a polynomial in $y_1, y_2, ..., y_n$. We still have an identity when (17) is differentiated in turn for each of the $y$'s. Hence

$$l_1 y_1^{l_1 - 1} \frac{\partial F}{\partial M_1} + l_2 y_1^{l_2 - 1} \frac{\partial F}{\partial M_2} + ... + l_n y_1^{l_n - 1} \frac{\partial F}{\partial M_n} \equiv 0,$$

$$l_1 y_2^{l_1 - 1} \frac{\partial F}{\partial M_1} + ... \equiv 0,$$

. . . . . .

$$l_1 y_n^{l_1 - 1} \frac{\partial F}{\partial M_1} + ... \equiv 0,$$

and so $$J \frac{\partial F}{\partial M_1} \equiv 0, \quad J \frac{\partial F}{\partial M_2} \equiv 0, \quad ..., \quad J \frac{\partial F}{\partial M_n} \equiv 0,$$

where $$J = \begin{vmatrix} y_1^{l_1 - 1} & y_1^{l_2 - 1} & . & . & . & y_1^{l_n - 1} \\ y_2^{l_1 - 1} & y_2^{l_2 - 1} & . & . & . & y_2^{l_n - 1} \\ . & . & . & . & . & . \\ y_n^{l_1 - 1} & y_n^{l_2 - 1} & . & . & . & y_n^{l_n - 1} \end{vmatrix}.$$

Some of the $\frac{\partial F}{\partial M_1}, \frac{\partial F}{\partial M_2}, ...$ may be identically congruent to zero; for example, $\frac{\partial F}{\partial M_1}$ is, if $F$ is independent of $M_1$; but they cannot all be thus identically congruent to zero since all the $A$'s in (17) are not congruent to zero. Further, $\frac{\partial F}{\partial M_1}$, say, is not identically congruent

to zero when expanded in powers of the $y$'s, for then we should have a polynomial congruence of the form (17) but of lower degree. This is impossible since $F$ is irreducible, and hence

$$J \equiv 0.$$

This cannot hold identically in $y_1, y_2, \ldots, y_n$ since no two of the $l$'s are congruent to each other mod $p-1$. For, on expressing $J$ as a polynomial in $y_n$, it clearly suffices to prove the theorem for $J$ with $n$ replaced by $n-1$, and hence by 2, for which the result is obvious.

The result (16) now follows, for there are $(2n)!/n!n!$ sets of $n$ variables which can be chosen as independent variables from among the $2n$ variables $x$ and $y$. The $n$ independent variables assume at most $p^n$ sets of values, and then there are at most $l_1 l_2 \ldots l_n$ sets of values for the other $n$ variables, their values not involving any other parameters than those given by the $n$ independent variables. There is, of course, the convention for the negative $l$ as explained in (16).

I close by taking this opportunity of acknowledging my great indebtedness to Mr. H. Davenport for his helpful criticism of my manuscript.

# SOME TRANSFORMATIONS OF GENERALIZED HYPERGEOMETRIC SERIES, AND CONTOUR-INTEGRALS OF BARNES'S TYPE

*By* W. N. BAILEY (*Manchester*)

[Received 17 December 1931]

**1.** During the last few years a number of papers have been written on transformations of generalized hypergeometric series with unit argument. Until 1923 the literature of the subject was scattered, but in that year Hardy* published his paper 'A chapter from Ramanujan's note-book' in which he gave an account and proofs of the results then known, most of which had been rediscovered by Ramanujan. The fundamental formula of that paper was Dougall's theorem† which can be written in the form

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & -n \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix};\right]
$$

$$
= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b-c-d)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}, \tag{1.1}
$$

where the parameters are connected by the relation

$$1+2a = b+c+d+e-n,$$

and $n$ is a positive integer.

Hardy's paper was followed by a number of papers by Whipple.‡ The series in Dougall's theorem belongs to a class to which Whipple gave the name 'well-poised', the series having parameters in pairs with the same sum. Whipple's most general result for well-poised series was§

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{matrix};\right]
$$

$$
= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)} \times
$$

$$
\times {}_4F_3\left[\begin{matrix} 1+a-c-d, & e, & f, & g \\ & e+f+g-a, & 1+a-c, & 1+a-d \end{matrix};\right] \tag{1.2}
$$

where the series on the right is terminating, i.e. one of the four numbers $e$, $f$, $g$, $1+a-c-d$, must be a negative integer.

* Hardy, **9**.

† Dougall, **8**, (6). See also Whipple, **12**, (8.2), where the formula is given in a form similar to the one used here. ‡ References are given below.

§ Whipple, **12**, (7.7). See also Whipple, **13**.

Later Whipple considered series which he termed 'nearly-poised', in which the parameters could be taken in pairs, each pair having the same sum with the exception of one pair. His most general results for nearly-poised series were*

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ & \kappa-b, & \kappa-c, & \kappa-d \end{matrix};\right]
$$

$$
= \frac{\Gamma(\kappa-b)\Gamma(\kappa-c)\Gamma(\kappa-d)\Gamma(\kappa-b-c-d)}{\Gamma(\kappa-c-d)\Gamma(\kappa-b-c)\Gamma(\kappa-b-d)\Gamma(\kappa)} \times
$$

$$
\times {}_5F_4\left[\begin{matrix} b, & c, & d, & \frac{1}{2}\kappa-\frac{1}{2}a, & \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ & \kappa-a, & \frac{1}{2}\kappa, & \frac{1}{2}+\frac{1}{2}\kappa, & b+c+d-\kappa+1 \end{matrix};\right] \quad (1.3)
$$

where $b$, $c$ or $d$ is a negative integer; and

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & -n; \\ 1+a-b, & 1+a-c, & w & \end{matrix}\right] = \frac{\Gamma(w)\Gamma(w-a+n)}{\Gamma(w+n)\Gamma(w-a)} \times \quad (1.4)
$$

$$
\times {}_5F_4\left[\begin{matrix} 1+a-w, & \frac{1}{2}(1+a), & \frac{1}{2}a, & 1+a-b-c, & -n \\ & 1+a-b, & 1+a-c, & 1+\frac{1}{2}(a-w-n), & \frac{1}{2}(1+a-w-n) \end{matrix};\right].
$$

In each case the series on the right is Saalschützian, i.e. the sum of the denominator parameters exceeds the sum of the numerator parameters by unity. In the series on the left of (1.3), the breakdown in the equality of sums of pairs of parameters occurs with the first pair, and in (1.4) it occurs with the last pair. We shall call these series nearly-poised series of the first and second kinds respectively.

In a paper† written some three or four years ago, I gave an algebraic method of obtaining transformations of terminating hypergeometric series, and obtained the natural successors to Whipple's theorems on nearly-poised series, as well as relations giving certain nearly-poised series of the types ${}_5F_4$ and ${}_6F_5$ in terms of well-poised series ${}_9F_8$. Probably the most interesting relation, however, was a formula connecting two well-poised series ${}_9F_8$, from which I later‡ deduced the form of (1.2) when the series were not terminating. This naturally suggests the problem of finding the relations corresponding to other known results when the series do not terminate.

In this paper I give a method which leads to such generalizations. Free use is made of contour-integrals of Barnes's type which I believe have only previously been used for this purpose§ in connexion with

* Whipple, **14**, (3.5) and (6.5). † Bailey, **1**. ‡ Bailey, **2**.

§ See Hardy, **9**, § 7; Whipple, **15**, § 2; Dougall, **8**, §§ 13 and 14. Dougall uses Barnes's integrals for series of higher order, but not in the same way as they are used here.

ordinary hypergeometric series and series of the type ${}_3F_2$. In the course of the work new integrals of this type are evaluated. The process of interchanging the order of integration, or the order of summation and integration, can be justified by standard methods similar to those used by Barnes,* and I consequently do not interrupt the argument to give these details or the conditions of convergence which can easily be supplied.

I assume only known theorems giving the sums of certain series ${}_3F_2$, but even these could be obtained by the same methods from Gauss's theorem for the sum of a ${}_2F_1$.

## 2. The fundamental integrals of Barnes

There are two fundamental integrals which were given by Barnes. The first appeared in his classical memoir† on the ordinary hypergeometric function, and is

$$-\frac{1}{2\pi i}\int \Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\beta_1-s)\Gamma(\beta_2-s)\,ds = \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)}, \quad (2.1)$$

where the path of integration is parallel to the imaginary axis, except that it is curved, if necessary, so that the decreasing sequences of poles lie to the left, and the increasing sequences to the right of the contour. All the contours used in this paper are of this type.

The second integral is‡

$$-\frac{1}{2\pi i}\int \frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3+s)\Gamma(1-\beta_1-s)\Gamma(-s)\,ds}{\Gamma(\beta_2+s)} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(1-\beta_1+\alpha_1)\Gamma(1-\beta_1+\alpha_2)\Gamma(1-\beta_1+\alpha_3)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_3)}, \quad (2.2)$$

provided that $\beta_1+\beta_2 = \alpha_1+\alpha_2+\alpha_3+1$, and was used by Barnes§ in connexion with the theory of Legendre functions. He proved it by using (2.1) and Gauss's theorem. We shall call this 'Barnes's second integral'. It will be noticed that the relation between the parameters is reminiscent of the relation involved in Saalschütz's theorem‖ for

* Barnes, **4** and **6**. See also Whittaker and Watson, **16**, §§ 14.5 and 14.53.

† Barnes, **4**, § 15; also Whittaker and Watson, **16**, § 14.52.

‡ Barnes, **5**, § 6. § Barnes, **6**, § 71.

‖ Saalschütz, **10**. See also Hardy, **9**, § 5. If the integral in (2.2) is evaluated in terms of hypergeometric series by considering the residues at poles on the right of the contour, we obtain a relation which reduces to Saalschütz's theorem when one of $\alpha_1$, $\alpha_2$, $\alpha_3$ is a negative integer.

the sum of a ${}_3F_2$. This similarity suggests some of the later results concerning contour-integrals.

## 3. Integrals representing well-poised series

From Barnes's second integral it is easily proved that

$$\frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)\Gamma(\alpha_3+n)}{\Gamma(\kappa-\alpha_1+n)\Gamma(\kappa-\alpha_2+n)\Gamma(\kappa-\alpha_3+n)}$$
$$= \frac{1}{\Gamma(\kappa-\alpha_2-\alpha_3)\Gamma(\kappa-\alpha_3-\alpha_1)\Gamma(\kappa-\alpha_1-\alpha_2)} \times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3+s)\Gamma(\kappa-\alpha_1-\alpha_2-\alpha_3-s)\Gamma(n-s)\,ds}{\Gamma(\kappa+n+s)}.$$

It follows by expansion and the interchange of the order of summation and integration* that

$$F\left[\begin{matrix}a, & \alpha_1, & \alpha_2, & \alpha_3, & \rho_1, \rho_2, \ldots, \rho_r; \\ & \kappa-\alpha_1, & \kappa-\alpha_2, & \kappa-\alpha_3, & \sigma_1, \sigma_2, \ldots, \sigma_r\end{matrix}\right]$$
$$= \frac{\Gamma(\kappa-\alpha_1)\Gamma(\kappa-\alpha_2)\Gamma(\kappa-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\kappa-\alpha_2-\alpha_3)\Gamma(\kappa-\alpha_1-\alpha_3)\Gamma(\kappa-\alpha_1-\alpha_2)} \times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3+s)\Gamma(\kappa-\alpha_1-\alpha_2-\alpha_3-s)\Gamma(-s)}{\Gamma(\kappa+s)} \times$$
$$\times F\left[\begin{matrix}a, \rho_1, \rho_2, \ldots, \rho_r, -s; \\ \sigma_1, \sigma_2, \ldots, \sigma_r, \kappa+s\end{matrix}\right] ds. \quad (3.1)$$

Thus, if we can sum the series on the right of (3.1) in terms of gamma functions, we can find an integral of Barnes's type representing the series on the left.

First take $r=1$, $\kappa = 1+a$, $\sigma_1 = 1+a-\rho_1$, and then we can sum the ${}_3F_2$ on the right by Dixon's theorem.† We thus find that

$${}_5F_4\left[\begin{matrix}a, & b, & c, & d, & e \\ & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}; \right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+\frac{1}{2}a)}{\left\{\begin{matrix}\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-c-d)\times \\ \times\Gamma(1+a-b-d)\Gamma(1+a-b-c)\Gamma(1+\frac{1}{2}a-e)\end{matrix}\right\}} \times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(1+\frac{1}{2}a-e+s)\times}{\times\Gamma(1+a-b-c-d-s)\Gamma(-s)\,ds} \quad (3.2)$$
$$\Gamma(1+\tfrac{1}{2}a+s)\Gamma(1+a-e+s)$$

* For the justification, cf. Barnes, **5**, § 3.

† Dixon, **7**. See also Watson, **11**, for another proof; also Hardy **9**, (3.2).

If $b = 1+\frac{1}{2}a$, the integral on the right can be evaluated by Barnes's second integral, and we obtain the formula*

$$
{}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix};\right]
$$

$$
= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}. \tag{3.3}
$$

Now use this formula on the right of (3.1) and we obtain

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & \end{matrix};\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\left\{\begin{matrix}\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-c-d)\times \\ \times\Gamma(1+a-b-d)\Gamma(1+a-b-c)\Gamma(1+a-e-f)\end{matrix}\right\}} \times
$$

$$
\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(1+a-e-f+s)\times}{\times\Gamma(1+a-b-c-d-s)\Gamma(-s)\,ds}. \tag{3.4}
$$

$$
\Gamma(1+a-e+s)\Gamma(1+a-f+s)
$$

If $f = -n$, where $n$ is a positive integer, and $1+2a = b+c+d+e-n$, the integral on the right can be evaluated by Barnes's second integral, and we obtain Dougall's theorem. We cannot use this theorem on the right of (3.1), and so this process comes to an end for well-poised series. The formula (3.2) is evidently the particular case of (3.4) when $f = \frac{1}{2}a$.

## 4. A transformation of well-poised series

We now evaluate the integral in (3.4) by considering the residues at the poles on the right of the contour, and we obtain the transformation

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & \end{matrix};\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-b-d)\Gamma(1+a-b-c)} \times
$$

$$
\times {}_4F_3\left[\begin{matrix} b, & c, & d, & 1+a-e-f; \\ b+c+d-a, & 1+a-e, & 1+a-f & \end{matrix}\right] +
$$

$$
+ \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-e-f)} \times
$$

* Whipple, **12**, (5.2), and Dougall, **8**, (9).

$$\times\frac{\Gamma(2+2a-b-c-d-e-f)\Gamma(b+c+d-a-1)}{\Gamma(2+2a-b-c-d-e)\Gamma(2+2a-b-c-d-f)}\times$$

$$\times{}_4F_3\left[\begin{matrix}1+a-c-d, 1+a-b-d, 1+a-b-c, 2+2a-b-c-d-e-f;\\ 2+2a-b-c-d-e, 2+2a-b-c-d-f, 2+a-b-c-d\end{matrix}\right]. \tag{4.1}$$

This is the generalization of Whipple's theorem on well-poised series which I previously obtained* by a less direct method.

## 5. Integrals representing nearly-poised series of the first kind

In (3.1) take $r=0$ and sum the ${}_2F_1$ on the right by Gauss's theorem, and we find that

$${}_4F_3\left[\begin{matrix}a, & b, & c, & d\\ & \kappa-b, & \kappa-c, & \kappa-d\end{matrix};\right]$$

$$=\frac{\Gamma(\kappa-b)\Gamma(\kappa-c)\Gamma(\kappa-d)}{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(\kappa-c-d)\Gamma(\kappa-b-d)\Gamma(\kappa-b-c)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(\kappa-a+2s)\times}{\times\Gamma(\kappa-b-c-d-s)\Gamma(-s)\,ds}\cdot \tag{5.1}$$

If $d=\frac{1}{2}+\frac{1}{2}\kappa$, $c=\frac{1}{2}\kappa$, the integral on the right can be evaluated by Barnes's second integral, and (changing $\kappa$ into $1+\kappa$) we find that

$${}_3F_2\left[\begin{matrix}a, & 1+\frac{1}{2}\kappa, & b\\ & \frac{1}{2}\kappa, & 1+\kappa-b\end{matrix};\right]$$

$$=\frac{\Gamma(\frac{1}{2}\kappa)\Gamma(1+\frac{1}{2}\kappa-\frac{1}{2}a)\Gamma(1+\kappa-b)\Gamma(\kappa-a-2b)}{\Gamma(1+\frac{1}{2}\kappa)\Gamma(\frac{1}{2}\kappa-\frac{1}{2}a)\Gamma(1+\kappa-a-b)\Gamma(\kappa-2b)}\cdot \tag{5.2}$$

Now use this result on the right of (3.1), and we obtain

$${}_5F_4\left[\begin{matrix}a, & 1+\frac{1}{2}\kappa, & b, & c, & d\\ & \frac{1}{2}\kappa, & 1+\kappa-b, & 1+\kappa-c, & 1+\kappa-d\end{matrix};\right]$$

$$=\frac{(\kappa-a)\Gamma(1+\kappa-b)\Gamma(1+\kappa-c)\Gamma(1+\kappa-d)}{\kappa\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+\kappa-c-d)\Gamma(1+\kappa-b-d)\Gamma(1+\kappa-b-c)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(\kappa-a+2s)\times}{\times\Gamma(1+\kappa-b-c-d-s)\Gamma(-s)\,ds}\cdot \tag{5.3}$$

## 6. Transformations of nearly-poised series of the first kind

We now evaluate the integrals on the right of (5.1) and (5.3) by considering the poles on the right of the contours, and so obtain the

* Bailey, **2**.

formulae

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ \kappa-b, & \kappa-c, & \kappa-d & \end{matrix};\right]
$$

$$
= \frac{\Gamma(\kappa-b)\Gamma(\kappa-c)\Gamma(\kappa-d)\Gamma(\kappa-b-c-d)}{\Gamma(\kappa-c-d)\Gamma(\kappa-b-d)\Gamma(\kappa-b-c)\Gamma(\kappa)} \times
$$

$$
\times {}_5F_4\left[\begin{matrix} b, & c, & d, & \frac{1}{2}(\kappa-a), & \frac{1}{2}(1+\kappa-a) \\ \kappa-a, & \frac{1}{2}\kappa, & \frac{1}{2}(\kappa+1), & 1-\kappa+b+c+d & \end{matrix};\right] +
$$

$$
+ \frac{\Gamma(\kappa-b)\Gamma(\kappa-c)\Gamma(\kappa-d)\Gamma(b+c+d-\kappa)\Gamma(3\kappa-a-2b-2c-2d)}{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(2\kappa-a-b-c-d)\Gamma(3\kappa-2b-2c-2d)} \times
$$

$$
\times {}_5F_4\left[\begin{matrix} \kappa-c-d, & \kappa-b-d, & \kappa-b-c, \\ & \frac{3}{2}\kappa-\frac{1}{2}a-b-c-d, & \frac{1}{2}+\frac{3}{2}\kappa-\frac{1}{2}a-b-c-d; \\ 1+\kappa-b-c-d, & 2\kappa-a-b-c-d, & \\ & \frac{3}{2}\kappa-b-c-d, & \frac{1}{2}+\frac{3}{2}\kappa-b-c-d \end{matrix}\right] \qquad (6.1)
$$

and

$$
{}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}\kappa, & b, & c, & d \\ & \frac{1}{2}\kappa, & 1+\kappa-b, & 1+\kappa-c, & 1+\kappa-d \end{matrix};\right]
$$

$$
= \frac{\Gamma(1+\kappa-b)\Gamma(1+\kappa-c)\Gamma(1+\kappa-d)\Gamma(1+\kappa-b-c-d)}{\Gamma(1+\kappa)\Gamma(1+\kappa-c-d)\Gamma(1+\kappa-b-d)\Gamma(1+\kappa-b-c)} \times
$$

$$
\times {}_5F_4\left[\begin{matrix} b, & c, & d, & \frac{1}{2}(\kappa-a), & \frac{1}{2}(1+\kappa-a); \\ b+c+d-\kappa, & 1+\kappa-a, & \frac{1}{2}\kappa, & \frac{1}{2}+\frac{1}{2}\kappa & \end{matrix}\right] +
$$

$$
+ \frac{\begin{matrix}(\kappa-a)\Gamma(1+\kappa-b)\Gamma(1+\kappa-c)\Gamma(1+\kappa-d)\times \\ \times\Gamma(b+c+d-\kappa-1)\Gamma(2+3\kappa-a-2b-2c-2d)\end{matrix}}{\kappa\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(2+2\kappa-a-b-c-d)\Gamma(2+3\kappa-2b-2c-2d)} \times
$$

$$
\times {}_5F_4\left[\begin{matrix} 1+\kappa-c-d, & 1+\kappa-b-d, & 1+\kappa-b-c, \\ & 1+\frac{3}{2}\kappa-\frac{1}{2}a-b-c-d, & \frac{3}{2}+\frac{3}{2}\kappa-\frac{1}{2}a-b-c-d; \\ 2+\kappa-b-c-d, & 2+2\kappa-a-b-c-d, & \\ & 1+\frac{3}{2}\kappa-b-c-d, & \frac{3}{2}+\frac{3}{2}\kappa-b-c-d \end{matrix}\right]. \qquad (6.2)
$$

The formula (6.1) is the required generalization of Whipple's transformation of nearly-poised series of the first kind, formula (1.3) above, and (6.2) is the generalization of a formula previously obtained by myself.*

## 7. The integral analogue of Dougall's theorem

As noted in § 2, there is a certain similarity between Barnes's second integral and Saalschütz's theorem, just as there is between (2.1) and Gauss's theorem. This suggests the possibility of evaluating

* Bailey, **1**, (8.51).

integrals of Barnes's type analogous to Dougall's theorem and the known theorems giving the sums of well-poised series which can be derived from Dougall's theorem.

First consider the integral

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\times \\ \times\Gamma(d+s)\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)},$$

which is analogous to a well-poised series ${}_5F_4$, the path of integration being, as usual, a line parallel to the imaginary axis, with any necessary curved parts to separate the increasing and decreasing sequences of poles.

Considering the residues at the poles on the right of the contour, we see that the integral can be expressed in terms of two well-poised series ${}_5F_4$ which can be summed by (3.3). We can thus evaluate the integral in terms of gamma functions, and we find after some reduction that

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\times \\ \times\Gamma(d+s)\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)} \\ =\frac{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(b+c-a)\Gamma(b+d-a)}{2\Gamma(1+a-c-d)\Gamma(b+c+d-a)}. \qquad (7.1)$$

Similarly, the more general integral

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(e+s)\Gamma(f+s)}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)\Gamma(1+a-e+s)}\times \\ \times\frac{\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(1+a-f+s)}$$

can be expressed in terms of two well-poised series ${}_7F_6$ which one might expect to be able to sum under suitable circumstances by Dougall's theorem. In this case, however, one of the parameters has to be a negative integer, and then the integral does not exist. We can, however, evaluate the integral in another way when there is no restriction that one parameter must be a negative integer.

From Barnes's second integral we have

$$\frac{\Gamma(d+s)\Gamma(e+s)\Gamma(f+s)}{\Gamma(1+a-d+s)\Gamma(1+a-e+s)\Gamma(1+a-f+s)} \\ =\frac{1}{\Gamma(1+a-e-f)\Gamma(1+a-d-f)\Gamma(1+a-d-e)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(d+t)\Gamma(e+t)\Gamma(f+t)\Gamma(1+a-d-e-f-t)\Gamma(s-t)\,dt}{\Gamma(1+a+s+t)},$$

the contour having been moved into a parallel position. Thus our integral is equal to*

$$-\frac{1}{2\pi i}\int\frac{\Gamma(d+t)\Gamma(e+t)\Gamma(f+t)\Gamma(1+a-d-e-f-t)\,dt}{\Gamma(1+a-e-f)\Gamma(1+a-d-f)\Gamma(1+a-d-e)}\times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\times}{\times\,\Gamma(s-t)\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a+s+t)}.$$

We can integrate with respect to $s$ by means of (7.1), and so obtain

$$-\frac{1}{2\pi i}\int\frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(e+s)}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)}\times$$
$$\times\frac{\Gamma(f+s)\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(1+a-e+s)\Gamma(1+a-f+s)}$$
$$=\frac{\Gamma(b)\Gamma(c)\Gamma(b+c-a)}{2\Gamma(1+a-e-f)\Gamma(1+a-d-f)\Gamma(1+a-d-e)}\times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(d+t)\Gamma(e+t)\Gamma(f+t)\Gamma(b-a-t)\times}{\times\,\Gamma(1+a-d-e-f-t)\Gamma(-t)\,dt}{\Gamma(1+a-c+t)\Gamma(b+c-a-t)}. \tag{7.2}$$

The integral on the right can be evaluated by Barnes's second integral when

$$1+2a=b+c+d+e+f,$$

and so we obtain the formula

$$-\frac{1}{2\pi i}\int\frac{\Gamma(a+s)\Gamma(1+\frac{1}{2}a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d+s)\Gamma(e+s)}{\Gamma(\frac{1}{2}a+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)}\times$$
$$\times\frac{\Gamma(f+s)\Gamma(b-a-s)\Gamma(-s)\,ds}{\Gamma(1+a-e+s)\Gamma(1+a-f+s)}$$
$$=\frac{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(b+c-a)\Gamma(b+d-a)\Gamma(b+e-a)\Gamma(b+f-a)}{\left\{\begin{array}{c}2\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)\times\\ \times\,\Gamma(1+a-c-f)\Gamma(1+a-d-f)\Gamma(1+a-e-f)\end{array}\right\}} \tag{7.3}$$

provided that

$$1+2a=b+c+d+e+f.$$

This is the analogue of Dougall's theorem.

From the relations given above we can, by considering residues at poles on the right of the contours, write down the corresponding

* For the justification of the interchange in the order of integration, cf. Whittaker and Watson, **16**, § 14.53.

relations involving hypergeometric series. Thus (7.2) gives a relation between two well-poised series of the type ${}_7F_6$ (either of which is of general type except for the second parameter) and three series ${}_4F_3$ which are Saalschützian in type except that they do not terminate. The result obtained from (7.3) is perhaps particularly interesting. It is

$$\frac{\Gamma(1+a)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(b-a)}{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}\times \tag{7.4}$$

$$\times {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & \end{matrix};\right]$$

$$= \frac{\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(b+c-a)\Gamma(b+d-a)\Gamma(b+e-a)\Gamma(b+f-a)}{\left\{\begin{matrix}\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)\times \\ \times\Gamma(1+a-c-f)\Gamma(1+a-d-f)\Gamma(1+a-e-f)\end{matrix}\right\}} -$$

$$-\frac{\Gamma(1+2b-a)\Gamma(b+c-a)\Gamma(b+d-a)\Gamma(b+e-a)\Gamma(b+f-a)\Gamma(a-b)}{\Gamma(1+b-c)\Gamma(1+b-d)\Gamma(1+b-e)\Gamma(1+b-f)}\times$$

$$\times {}_7F_6\left[\begin{matrix} 2b-a, & 1+b-\frac{1}{2}a, & b, & b+c-a, & b+d-a, & b+e-a, & b+f-a; \\ & b-\frac{1}{2}a, & 1+b-a, & 1+b-c, & 1+b-d, & 1+b-e, & 1+b-f \end{matrix}\right]$$

where $1+2a = b+c+d+e+f$, and is the form assumed by Dougall's theorem when we remove the restriction that one of the parameters must be a negative integer. The formula (4.1) gives a different generalization of Dougall's theorem, but here we have only one series on the right instead of two.

## 8. A method of obtaining transformations of integrals of Barnes's type

In (7.3) put $k$, $k+c-a$, $k+d-a$, $k+e-a$, $a$, $a+t$ for $a$, $c$, $d$, $e$, $k$, $b$, where $k = 1+2a-c-d-e$, and we find that

$$\frac{\Gamma(a+t)\Gamma(c+t)\Gamma(d+t)\Gamma(e+t)\Gamma(-t)}{\Gamma(1+a-c+t)\Gamma(1+a-d+t)\Gamma(1+a-e+t)}$$

$$= \frac{2\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(a-k)\Gamma(k+c-a)\Gamma(k+d-a)\Gamma(k+e-a)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(k+s)\Gamma(1+\frac{1}{2}k+s)\times}{\times\Gamma(k+c-a+s)\Gamma(k+d-a+s)\Gamma(k+e-a+s)}\times$$

$$\frac{}{\Gamma(\frac{1}{2}k+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)\Gamma(1+a-e+s)}\times$$

$$\times\frac{\Gamma(-t+s)\Gamma(a+t+s)\Gamma(a-k+t-s)\Gamma(-s)\,ds}{\Gamma(1+k+t+s)}.$$

Now multiply by $\dfrac{\Gamma(\rho_1+t)\Gamma(\rho_2+t)\Gamma(b-a-t)}{\Gamma(\sigma_1+t)}$,

integrate with respect to $t$,* and then put $t+s$ for $t$ on the right. We thus find that

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+t)\Gamma(c+t)\Gamma(d+t)\Gamma(e+t)\times}{\times\Gamma(\rho_1+t)\Gamma(\rho_2+t)\Gamma(b-a-t)\Gamma(-t)\,dt}{\Gamma(1+a-c+t)\Gamma(1+a-d+t)\Gamma(1+a-e+t)\Gamma(\sigma_1+t)}$$

$$=\frac{2\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(a-k)\Gamma(k+c-a)\Gamma(k+d-a)\Gamma(k+e-a)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(k+s)\Gamma(1+\frac{1}{2}k+s)\Gamma(k+c-a+s)\times}{\times\Gamma(k+d-a+s)\Gamma(k+e-a+s)\Gamma(-s)\,ds}{\Gamma(\frac{1}{2}k+s)\Gamma(1+a-c+s)\Gamma(1+a-d+s)\Gamma(1+a-e+s)}\times$$

$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(a+2s+t)\Gamma(a-k+t)\Gamma(\rho_1+s+t)\times}{\times\Gamma(\rho_2+s+t)\Gamma(b-a-s-t)\Gamma(-t)\,dt}{\Gamma(1+k+2s+t)\Gamma(\sigma_1+s+t)}, \quad (8.1)$$

where $k = 1+2a-c-d-e$.

In this formula there may be any number of the quantities $\rho$ and $\sigma$. If we can integrate with respect to $t$ on the right, we obtain a relation between two integrals of Barnes's type.

## 9. An integral related to well-poised series

In (8.1) take $\rho_1 = 1+\frac{1}{2}a$, $\rho_2 = f$, $\rho_3 = g$, $\rho_4 = h$, $\rho_5 = b$, $\sigma_1 = \frac{1}{2}a$, $\sigma_2 = 1+a-f$, $\sigma_3 = 1+a-g$, $\sigma_4 = 1+a-h$. Then we can integrate with respect to $t$ on the right by means of (7.3), if

$$2+3a = b+c+d+e+f+g+h.$$

We thus find that

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+t)\Gamma(1+\frac{1}{2}a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(e+t)}{\Gamma(\frac{1}{2}a+t)\Gamma(1+a-c+t)\Gamma(1+a-d+t)\Gamma(1+a-e+t)}\times$$

$$\times\frac{\Gamma(f+t)\Gamma(g+t)\Gamma(h+t)\Gamma(b-a-t)\Gamma(-t)\,dt}{\Gamma(1+a-f+t)\Gamma(1+a-g+t)\Gamma(1+a-h+t)}$$

$$=\frac{\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f+b-a)\Gamma(g+b-a)\Gamma(h+b-a)}{\left\{\begin{matrix}\Gamma(k+c-a)\Gamma(k+d-a)\Gamma(k+e-a)\times\\ \times\Gamma(1+a-g-h)\Gamma(1+a-f-h)\Gamma(1+a-f-g)\end{matrix}\right\}}\times$$

* For the justification, cf. Whittaker and Watson, **16**, § 14.53. The lower bound of the distance between the $s$-contour and the $t$-contour is supposed to be positive (not zero).

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(k+s)\Gamma(1+\frac{1}{2}k+s)\Gamma(b+s)\times}{\times\Gamma(k+c-a+s)\Gamma(k+d-a+s)}\times$$
$$\times\frac{\Gamma(k+e-a+s)\Gamma(f+s)\Gamma(g+s)\Gamma(h+s)\Gamma(b-k-s)\Gamma(-s)\,ds}{\Gamma(1+k-f+s)\Gamma(1+k-g+s)\Gamma(1+k-h+s)}, \quad (9.1)$$

where $k = 1+2a-c-d-e$, and the parameters are connected by the relation

$$2+3a = b+c+d+e+f+g+h.$$

From (9.1), by considering the residues at the poles on the right of the contours, we can write down a relation involving four well-poised series of the type ${}_9F_8$, the numerator parameters being given by the scheme

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $1+\frac{1}{2}a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
| $2b-a$ | $1+b-\frac{1}{2}a$ | $b$ | $b+c-a$ | $b+d-a$ | $b+e-a$ | $b+f-a$ | $b+g-a$ | $b+h-a$ |
| $k$ | $1+\frac{1}{2}k$ | $b$ | $c+k-a$ | $d+k-a$ | $e+k-a$ | $f$ | $g$ | $h$ |
| $2b-k$ | $1+b-\frac{1}{2}k$ | $b$ | $b+c-a$ | $b+d-a$ | $b+e-a$ | $b+f-k$ | $b+g-k$ | $b+h-k$ |

The parameters of the second series are obtained from those of the first series by the addition of $b-a$ to each, the parameter so obtained from $b$ becoming the first parameter of the second series. We shall say that the two series are 'complementary with respect to the parameter $b$'.

In the above relation, any one of the series ${}_9F_8$ is of general type except for the second parameter and the restriction that the sum of the denominator parameters exceeds the sum of the numerator parameters by two. In the case when $f$, $g$ or $h$ is a negative integer, this relation reduces to the relation between two terminating well-poised series ${}_9F_8$ from which I previously deduced* the generalization of Whipple's theorem on well-poised series.

## 10. Integrals related to Saalschützian nearly-poised series

Now in the formula (8.1) take $\rho_1 = f$, $b = 1+2k-a-f$, so that we can integrate with respect to $t$ on the right by means of Barnes's second integral. We are thus led to the formula

$$-\frac{1}{2\pi i}\int\frac{\Gamma(a+t)\Gamma(c+t)\Gamma(d+t)\Gamma(e+t)\times}{\times\Gamma(f+t)\Gamma(1+2k-2a-f-t)\Gamma(-t)\,dt}$$
$$=\frac{2\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(1+2k-2a)}{\Gamma(k+c-a)\Gamma(k+d-a)\Gamma(k+e-a)\Gamma(1+k-a)}\times$$

* Bailey, **1**, (8.6), and Bailey, **2**.

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(k+s)\Gamma(1+\frac{1}{2}k+s)\Gamma(a+2s)\Gamma(f+s)\Gamma(k+c-a+s)}{\Gamma(\frac{1}{2}k+s)\Gamma(1+2k-a+2s)\Gamma(1+k-f+s)}\times$$
$$\times\frac{\Gamma(k+d-a+s)\Gamma(k+e-a+s)\times}{\times\Gamma(1+2k-a-f+s)\Gamma(1+k-a-f-s)\Gamma(-s)\,ds}{\Gamma(1+a-c+s)\Gamma(1+a-d+s)\Gamma(1+a-e+s)}, \quad (10.1)$$

where $k = 1+2a-c-d-e$.

From this formula we find, in the usual way, a relation in which there are two nearly-poised series ${}_5F_4$ on the left and two well-poised series ${}_9F_8$ on the right.

The nearly-poised series are

$${}_5F_4\left[\begin{matrix}a, & c, & d, & e, & f \\ & 1+a-c, & 1+a-d, & 1+a-e, & 2a+f-2k\end{matrix};\right]$$

and

$${}_5F_4\left[\begin{matrix}1+2k-2a, & 1+2k-2a-f+c, & 1+2k-2a-f+d, \\ & & 1+2k-2a-f+e, & 1+2k-a-f; \\ 2+2k-a-f-c, & 2+2k-a-f-d, & \\ & & 2+2k-a-f-e, & 2+2k-2a-f\end{matrix}\right]$$

while one series ${}_9F_8$ has, in the numerator, the parameters $k$, $1+\frac{1}{2}k$, $\frac{1}{2}a$, $\frac{1}{2}+\frac{1}{2}a$, $f$, $k+c-a$, $k+d-a$, $k+e-a$, $1+2k-a-f$, and the other ${}_9F_8$ is complementary to this with respect to the parameter $1+2k-a-f$. The nearly-poised series are Saalschützian in type except that they do not terminate. One of them is a nearly-poised series of the first kind and the other is of the second kind. In the particular case when $f$ is a negative integer, this relation reduces to a transformation of a Saalschützian nearly-poised series ${}_5F_4$ of the second kind into a well-poised ${}_9F_8$,* while, if one of the parameters $2+2k-2a+c-f$, $2+2k-2a+d-f$, $2+2k-2a+e-f$ is a negative integer, the relation reduces to a transformation of a Saalschützian nearly-poised series ${}_5F_4$ of the first kind into a well-poised ${}_9F_8$.† The transformation obtained from (10.1) is thus the generalization of two known formulae.

As a particular case of (10.1) take $d = 1+\frac{1}{2}a$, $e = \frac{1}{2}+\frac{1}{2}a$, so that $k = a-c-\frac{1}{2}$, and the integral on the right can be evaluated by (7.1). Replacing $f$ by $d$, we obtain the formula

$$-\frac{1}{2\pi i}\int\frac{\Gamma(a+t)\Gamma(1+\frac{1}{2}a+t)\Gamma(c+t)\Gamma(d+t)\Gamma(-2c-d-t)\Gamma(-t)\,dt}{\Gamma(\frac{1}{2}a+t)\Gamma(1+a-c+t)}$$
$$=\frac{\Gamma(1+a)\Gamma(c)\Gamma(d)\Gamma(-2c)\Gamma(a-2c-d)\Gamma(-c-d)}{2\Gamma(a-2c)\Gamma(1+a-c-d)\Gamma(-c)}. \quad (10.2)$$

* Bailey, **1**, (8.1). † Bailey, **1**, (8.11).

Now in (8.1) take $\rho_1 = 1+\frac{1}{2}a$, $\rho_2 = f$, $\sigma_1 = \frac{1}{2}a$. The integration on the right with respect to $t$ can be performed by (10.2) provided that $b = 2k-a-f$, and we are led to

$$-\frac{1}{2\pi i}\int \frac{\Gamma(a+t)\Gamma(1+\frac{1}{2}a+t)\Gamma(c+t)\Gamma(d+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(\frac{1}{2}a+t)\Gamma(1+a-c+t)\Gamma(1+a-d+t)}\times$$
$$\times\frac{\Gamma(2k-2a-f-t)\Gamma(-t)\,dt}{\Gamma(1+a-e+t)}$$
$$=\frac{\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(2k-2a)}{\Gamma(k+c-a)\Gamma(k+d-a)\Gamma(k+e-a)\Gamma(k-a)}\times$$
$$\times\left(-\frac{1}{2\pi i}\right)\int \frac{\Gamma(k+s)\Gamma(1+\frac{1}{2}k+s)\Gamma(k+c-a+s)\times}{\times\Gamma(k+d-a+s)\Gamma(k+e-a+s)}\times$$
$$\times\frac{\Gamma(1+a+2s)\Gamma(f+s)\Gamma(2k-a-f+s)\Gamma(k-a-f-s)\Gamma(-s)\,ds}{\Gamma(2k-a+2s)\Gamma(1+k-f+s)} \qquad (10.3)$$

where $k = 1+2a-c-d-e$.

By considering the residues at poles on the right of the contours, we obtain a relation involving two nearly-poised series ${}_6F_5$ (Saalschützian in type except that they do not terminate) and two well-poised ${}_9F_8$ which are complementary with respect to the parameter $2k-a-f$. One of the nearly-poised series is of the first kind and the other of the second kind. When $f$ is a negative integer the relation reduces to the transformation of a Saalschützian nearly-poised ${}_6F_5$ of the second kind into a well-poised ${}_9F_8$;* and when one of the parameters $2k-2a+c-f$, $2k-2a+d-f$, $2k-2a+e-f$ is a negative integer, the relation reduces to a similar transformation of a nearly-poised series of the first kind.†

**11.** Similar methods give generalizations of other known results. For example, results of a type exactly similar to those of the last paragraph are true for nearly-poised series in which the sum of the denominator parameters exceeds the sum of the numerator parameters by two,‡ which is the condition satisfied by the parameters in Dougall's theorem.

The formula (1.4) and its successor§ are rather more troublesome to generalize, and the final result was unexpected. The formulae obtained involve five series instead of three or four as previously

* Bailey, **1**, (8.2). † Bailey, **1**, (8.21).

‡ Cf. Bailey, **1**, (8.3), (8.31), (8.4), and (8.41). § Bailey, **1**, (8.5).

obtained.* In each case two of the series are nearly-poised and of the second kind, one is nearly-poised and of the first kind, and the other two are Saalschützian in type. In the course of these investigations some integrals of Barnes's type are evaluated analogous to known sums of hypergeometric series.† Considerations of space, however, prevent these results being given in detail.

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* See Bailey, **1**, § 8, formula (A), and Bailey, **3**, (6.2).

† Except for the relation obtained from (7.2) where five series are involved.

# ON CALCULATIONS OF POLYHEDRAL MEAN VALUES

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SPHERICAL harmonics admitting the planes of symmetry of a regular solid are generated by certain invariant differential operators. A discreet use of this method of harmonic analysis may shorten the calculation of a type of symmetric function, whose evaluation by methods of pure algebra may be prohibitively laborious.

If we consider a system of planes in three dimensions, which are the planes of symmetry of a regular solid, any point $P$ can be transformed by repeated reflection into $N$ positions $P_1$, $P_2$,..., $P_N$ (including $P$ itself). For the tetrahedral configuration $N = 24$; for the octahedral and hexahedral (cube) configuration $N = 48$; and for the icosahedral and dodecahedral configuration $N = 120$. From any given well-defined function of position $f(P)$ we can form a symmetric function, invariant for the transformations of the group,

$$f^*(P) = \frac{1}{N} \sum f(P_r), \tag{1}$$

which gives a composite picture of the average distribution of the values of $f(P)$ in $N$ sectors into which the space around the origin is divided by the system of planes. It is required to evaluate $f^*(P)$.

For the tetrahedron, octahedron, and cube we may take the coordinate planes parallel to the faces of the cube; this fixes the orientation of the complementary octahedron and of the regular tetrahedra formed by alternate corners of the cube. The coordinates of a tetrahedral set of 24 points are obtained by making all permutations and all changes of sign in the coordinates, which leave the product $xyz$ invariant. On removing this restriction, we obtain the octahedral set of 48 points. If $f(P)$ is a power-series in $(x, y, z)$, terms whose signs are changed by these substitutions can be erased, and we have only to form the symmetric sum of six reduced forms.

But this direct procedure breaks down in the icosahedral system, owing to the obliqueness and complexity of the planes of symmetry. There are, nevertheless, certain obvious preliminary simplifications which may be made before the main problem is attacked. In the

standard configuration, the axis $Oz$ lies in a diameter of the icosahedron, and the plane $Ozx$ contains an edge in the positive quadrant. By symmetry about the origin and about the plane $y = 0$, we may replace the given function by the mean value

$$f^{(1)}(x, y, z) \equiv \tfrac{1}{4}[f(x, y, z)+f(x, -y, z)+f(-x, -y, -z)+f(-x, y, -z)], \tag{2}$$

which is even in $y$ separately and in $(x, y, z)$ jointly.

It will be a matter of choice whether we consider at this stage the effect of rotation about $Oz$ through $\frac{2}{5}\pi$, or whether we postpone this step. If we introduce cylindrical coordinates, and expand each term as a trigonometric polynomial in the azimuth $\phi$, the only terms which need be retained are those involving the azimuthal factors $\cos 5k\phi$. The sum of all other types of terms, when $\phi$ is replaced by $\phi+\frac{2}{5}s\pi$, gives a zero mean value. We shall suppose this reduction effected now, so that the function considered has the properties

$$\begin{aligned} f^{(2)}(x, y, z) &\equiv f^{(2)}(-x, -y, -z) \equiv f^{(2)}(x, -y, z) \\ &\equiv f^{(2)}(x\cos\tfrac{2}{5}\pi - y\sin\tfrac{2}{5}\pi, x\sin\tfrac{2}{5}\pi + y\cos\tfrac{2}{5}\pi, z). \end{aligned} \tag{3}$$

A function satisfying (3) is invariant for the sub-group of operations of reflection in the plane $Ozx$, reflection in the origin, and rotation about $Oz$ through a multiple of $\frac{2}{5}\pi$. In order to apply our method, it becomes necessary to express the function in terms of harmonic polynomials, which may be multiplied by any powers of $r$, positive or negative. To fix ideas and to avoid irrelevant difficulties, we suppose the given function analytic at the origin, so that it can be expanded as a convergent multiple power-series. The only groups of terms which are left after the preliminary reduction are of even order. These must now be expanded by classical methods in the form†

$$F_n(x, y, z) = H_n(x, y, z)+r^2H_{n-2}(x, y, z)+\dots. \tag{4}$$

That this is always possible was proved by Gauss, and there is a formula due to Dougall which gives the terms explicitly. The polynomials $H_k$ are homogeneous and satisfy Laplace's equation.

If, instead of an analytic function defined in the space about the origin, we are given an arbitrary function of position on the surface of a unit sphere, the same principles must be applied. We require to expand the given function as a series of surface harmonics, but, by symmetry, we need not trouble to evaluate the coefficients of any

† Hobson, p. 148.

terms, except those of the type $\cos 5k\phi P_{2n}^{5k}(\cos\theta)$. The number of integrations required will be about one-tenth of those needed to obtain the complete expansion, with the same degree of accuracy.

Any typical solid harmonic of even order $H_{2n}(x,y,z)$ may be treated as follows. By Hobson's theorem on differentiation,† we obtain from any homogeneous polynomial of order $n$ the identity

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{1}{r}$$
$$=\frac{(-)^n 2n!}{2^n n! r^{2n+1}}\left\{1-\frac{r^2\nabla^2}{2.2n-1}+\frac{r^4\nabla^4}{2.4.2n-1.2n-3}-\ldots\right\}f_n(x,y,z), \quad (5)$$

which gives for a solid harmonic simply

$$H_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{1}{r}=\frac{(-)^n 2n!}{2^n n! r^{2n+1}}H_n(x,y,z). \quad (6)$$

We thus pass at once from the solid harmonic to its generating operator. It is convenient to adopt the compact notation of Hilbert and Courant for the operators

$$\xi=\frac{\partial}{\partial x}, \qquad \eta=\frac{\partial}{\partial y}, \qquad \zeta=\frac{\partial}{\partial z}; \quad (7)$$

but in order to preserve the advantages of using Maxwell's variables $x\pm iy$, we shall also put

$$\lambda=\xi-i\eta, \qquad \mu=\xi+i\eta. \quad (8)$$

Since we operate only on $1/r$ and its derivatives, which satisfy Laplace's equation, we have

$$\xi^2+\eta^2+\zeta^2=0, \quad (9)$$

or

$$\lambda\mu+\zeta^2=0. \quad (10)$$

The generating operator is to be written in terms of $(\lambda,\mu,\zeta)$; if (10) is used to eliminate the product $\lambda\mu$, and if the preliminary symmetrization has reduced the function to a form satisfying (3), we have

$$H_{2n}=\sum a_{5k}(\lambda^{5k}+\mu^{5k})\zeta^{2n-5k}. \quad (11)$$

The operators $(\lambda,\mu,\zeta)$ are comparable with homogeneous variables, and combine among themselves as though they were replaced by $(t,-1/t,1)$.

As a compensation for the labour of constructing the expansion in spherical harmonics, we obtain advantages equivalent to the use

† Hobson, p. 127.

of binary, instead of ternary, forms. When the icosahedral transformations have been carried out on the operators, we can pass back to the original variables at the last stage of the work.

We now introduce the two fundamental invariant operators, which are unchanged by all the icosahedral transformations, namely,

$$\left.\begin{aligned} P &\equiv (\lambda^5+\mu^5)\zeta+11\zeta^6, \\ Q &\equiv \lambda^{10}+\mu^{10}-228(\lambda^5+\mu^5)\zeta^5+494\zeta^{10}, \end{aligned}\right\} \tag{12}$$

and we may note also the identities

$$\lambda^{10}+\mu^{10}-228\zeta^4 P+3002\zeta^{10} = Q, \tag{13}$$

$$3125\zeta^{12}-250P\zeta^6-Q\zeta^2+P^2 = 0. \tag{14}$$

We can now eliminate from (11) the operators $\lambda$, $\mu$, and we obtain an expression identical with $H$, but involving $P$, $Q$, $\zeta^2$. If we remove, by means of (14), all even powers of $\zeta$ equal to or higher than the twelfth, the result is expressible as a polynomial in $\zeta$ containing only even powers up to the tenth, with coefficients which are polynomials in $P$, $Q$. This expression is unique. For if we had

$$\sum_{r=0}^{5} I_{2r}\zeta^{2r} \equiv \sum_{r=0}^{5} J_{2r}\zeta^{2r}, \tag{15}$$

where the coefficients $I_{2r}$, $J_{2r}$ are invariants, we should find, on making all possible transformations, the same identity, except that $\zeta$ would be replaced by another operator $\partial/\partial h$. These identities could not hold unless the coefficients were the same.

It is not necessary to construct all the transforms of the operator $H$. For it is known that the twelve operators $\Phi_s \equiv \partial/\partial h_s$ into which $\zeta$ can be changed all satisfy the identity

$$3125\Phi^{12}-250P\Phi^6-Q\Phi^2+P^2 = 0. \tag{16}$$

Sums of powers of the operators are calculated formally like the sums of the roots of an equation. The sums of degree not exceeding 20 which do not vanish identically are

$$\left.\begin{aligned} &S_0 = 12, \quad S_6 = \frac{12}{25}P, \quad S_{10} = \frac{2}{625}Q, \quad S_{12} = \frac{108}{3125}P^2, \\ &S_{16} = 32PQ/5^7, \quad S_{18} = 204P^3/5^7, \quad S_{20} = 2Q^2/5^9, \ldots \end{aligned}\right\} \tag{17}$$

from which the 'mean value' of the operators can be calculated.

**Example (1).** Let $f(P)$ be a homogeneous polynomial of degree 4, say $f_4(x,y,z) \equiv \sum a_{rst}x^r y^s z^t$. This can be expanded in the form $f_4 = H_4+r^2H_2+r^4H_0$. But it is unnecessary to calculate $H_4$ and $H_2$, since the mean of their transforms must vanish, because there are

no symmetric polynomials of degree intermediate between 0 and 6. Thus the mean value is simply the term $r^4 H_0$, which has actually spherical symmetry. We find on operating twice with Laplace's operator

$$120 H_0 = \nabla^4 f_4 = 24 \sum a_{400} + 8 \sum a_{220},$$

$$H_0 = \tfrac{1}{5} \sum a_{400} + \tfrac{1}{15} \sum a_{220}.$$

**Example (2).** Let $f(P)$ be the most general harmonic polynomial of degree 20. Writing the operator in terms of $(\lambda, \mu, \zeta)$, the preliminary reduction enables us to reject every term except the set

$$H^{(1)} = \sum_{k=0}^{4} a_{5k} (\lambda^{5k} + \mu^{5k}) \zeta^{20-5k}.$$

This is the same as the form

$$\begin{aligned} H^{(1)} = a_0 \zeta^{20} + a_5 (P\zeta^{14} - 11\zeta^{20}) + a_{10} (P^2 \zeta^8 - 22 P\zeta^{14} + 123\zeta^{20}) + \\ + a_{15} (P^3 \zeta^2 - 33 P^2 \zeta^8 + 336 P\zeta^4 - 1364 \zeta^{20}) + \\ + a_{20} (Q^2 + 456 PQ\zeta^4 + 51984 P^2 \zeta^8 - \\ - 6004 Q\zeta^{10} - 1368912 P\zeta^{14} + 9012002 \zeta^{20}). \end{aligned}$$

From (17), we know that the symmetric sums corresponding to $\zeta^2$, $\zeta^4$, $\zeta^8$, $\zeta^{14}$ vanish, so that the only significant terms are

$$\begin{aligned} H^{(2)} = (a_0 - 11 a_5 + 123 a_{10} - 1364 a_{15} + 9012002 a_{20}) \zeta^{20} - \\ - 6004 a_{20} Q\zeta^{10} + a_{20} Q^2, \end{aligned}$$

whose 'mean value' is given by

$$6.5^9 . H^* = (a_0 - 11 a_5 + 123 a_{10} - 1364 a_{15} + 1968252 a_{20}) Q^2.$$

The same methods are legitimate, but much less useful, in the tetrahedral system of transformations, whose invariant operators are

$$L = \xi\eta\zeta, \qquad M = \eta^2\zeta^2 + \zeta^2\xi^2 + \xi^2\eta^2. \tag{18}$$

We may note in conclusion that the system of operators $P^a Q^b$ or $L^a M^b$ generate linearly independent combinations of tesseral harmonics. For if $n = 6a + 10b$, the highest powers of $\lambda$ and $\mu$ in $P^a Q^b$ are in $(\lambda^{5a+10b} + \mu^{5a+10b})\zeta^a$, or $(\lambda^{n-a} + \mu^{n-a})\zeta^a$, giving a multiple of $\cos(n-a)\phi P_n^{n-a}(\cos\theta)$. Since two pairs of values $(a, b)$ corresponding to the same $n$ cannot give the same highest term in $\cos(n-a)\phi$, the linear independence of each system is easily proved. Unfortunately, the two systems are not 'orthogonal'. Functions of different orders are of course orthogonal over the sphere. The earliest pairs of

harmonics of equal order are $(L^4, M^3)$ and $(P^5, Q^3)$ of order 12 and 30 respectively. I have verified that the former pair is not orthogonal, by expanding the operators in the canonical form

$$\sum a_{4k}(\lambda^{4k}+\mu^{4k})\zeta^{12-4k}.$$

Among the functions $\lambda^s\zeta^{n-s}(1/r)$, $\mu^s\zeta^{n-s}(1/r)$, $\zeta^n(1/r)$, all pairs are orthogonal, except those of the type

$$\int \left(\lambda^s\zeta^{n-s}\frac{1}{r}\right)\left(\mu^s\zeta^{n-s}\frac{1}{r}\right) dS = \frac{4\pi}{2n+1}\frac{(n+s)!(n-s)!}{R^{2n}}, \tag{19}$$

and it is not difficult to show that the mutual integral of the harmonics corresponding to $(L^4, M^3)$ does not vanish. It is theoretically possible to construct a complete orthogonal system out of the given systems, but the formulae for the coefficients are not likely to be simple or elegant.

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# SOME SINGULAR MODULI (II)

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THE set of singular moduli postponed from my previous paper* consists of $G_{121}$, $G_{169}$, $G_{225}$, $G_{325}$, $G_{363}$, $G_{1225}$, $g_{62}$, $g_{94}$, $g_{98}$, $g_{158}$; I give a discussion of them in this paper, and I include $g_{254}$ in addition, since it occurs naturally in the investigation of $g_{94}$ and $g_{158}$.

For convenience of reference I recall that, in the ordinary notation of elliptic functions,

$$q = e^{-\pi\sqrt{n}},$$

$$G_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}(1+q)(1+q^3)(1+q^5)\ldots = (2kk')^{-\frac{1}{24}},$$

$$g_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}(1-q)(1-q^3)(1-q^5)\ldots = \left(\frac{k'^2}{2k}\right)^{\frac{1}{12}}.$$

My endeavour has been to reconstruct the reasoning by which Ramanujan obtained these singular moduli, and I shall consequently use freely the modular equations contained in his note-book; some of these equations are new and others are in novel forms. I shall quote the equations in the notation used by Ramanujan; he works with the *squares* of the two moduli, which he calls $\alpha$ and $\beta$; and his phrase that '$\beta$ is of the $n$th degree in $\alpha$' means that the moduli $\sqrt{\alpha}$ and $\sqrt{\beta}$ are connected by the modular equation of order $n$, so that the parameters $q$ and $q_1$ associated with the moduli $\sqrt{\alpha}$ and $\sqrt{\beta}$ respectively are connected by the equation

$$q_1 = q^n.$$

Although certainty in such matters is evidently impossible, I think it very unlikely that Ramanujan obtained any of the singular moduli discussed in this paper by means of the empirical process used in my first paper; but he may have succeeded in abbreviating some of the calculations (e.g. in the cases of $G_{169}$ and $G_{1225}$) in ways which I have not rediscovered.

As in my first paper, at the beginning of the investigation of each value of $G_n$ and $g_n$, I have noted the generic character and the class-number of the determinant $-n$, as given by Gauss, *Werke*, 2, 450–74.

* See above, pp. 81–98.

(xvii)* $\mathbf{n = 121 = 11^2}$. (G. II, class-number 3.)

The modular equation of order 11 was known to Ramanujan in the form†

$$\sqrt[4]{\{\alpha\beta\}}+\sqrt[4]{\{(1-\alpha)(1-\beta)\}}+2\sqrt[12]{\{16\alpha\beta(1-\alpha)(1-\beta)\}} = 1.$$

If we put $16\alpha\beta(1-\alpha)(1-\beta) = t^{12}$, we have

$$\sqrt[4]{\{\alpha\beta\}}+\sqrt[4]{\{(1-\alpha)(1-\beta)\}} = 1-2t,$$

$$\sqrt[4]{\{\alpha\beta\}}.\sqrt[4]{\{(1-\alpha)(1-\beta)\}} = \tfrac{1}{2}t^3,$$

and hence, by solving a quadratic equation,‡

$$2\sqrt[4]{\{\alpha\beta\}} = 1-2t-R, \qquad 2\sqrt[4]{\{(1-\alpha)(1-\beta)\}} = 1-2t+R,$$

where $$R^2 = 1-4t+4t^2-2t^3.$$

By taking the fourth power of each of these equations, we have

$$2\alpha\beta = A-BR, \qquad 2(1-\alpha)(1-\beta) = A+BR,$$

where $$A = 1-8t+24t^2-34t^3+24t^4-8t^5+\tfrac{1}{2}t^6,$$

$$B = (1-2t)(1-4t+4t^2-t^3),$$

$$A^2-B^2R^2 = \tfrac{1}{4}t^{12}.$$

Now take $\alpha = \frac{1}{2}$ so that $4\beta(1-\beta) = G_{121}^{-24}$; then

$$\beta = A-BR, \qquad 1-\beta = A+BR,$$

so that $2A = 1$, while $t = G_{121}^{-2}$.

If we omit the suffix 121 for brevity, we deduce the equation

$$G^{12}-16G^{10}+48G^8-68G^6+48G^4-16G^2+1 = 0$$

as an immediate consequence of the equation $2A = 1$.

When we write $x$ in place of $G+G^{-1}$, this reciprocal equation reduces to

$$x^6-22x^4+121x^2-198 = 0.$$

From the $q$-product it follows that

$$G_{121} = 3{\cdot}54880\,1135, \qquad x = 3{\cdot}83058\,6437;$$

and so we deduce from the sextic equation that

$$x^3-11x = +3\sqrt{22}.$$

When we calculate the real root of this cubic equation, we obtain Ramanujan's first result

$$G_{121}+\frac{1}{G_{121}} = \left(\frac{11}{2}\right)^{\frac{1}{6}}\left\{\left(3+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(3-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\}.$$

* The numbering of the investigations is consecutive with the numbering in the first paper.

† See Schröter, *Journal für Math.* 58 (1861), 378–9.

‡ The positive value of $R$ is taken when $\alpha < 1-\beta$.

Again, it can be deduced immediately that

$$G^3-8G-3\sqrt{22}-8G^{-1}+G^{-3}=0,$$

either from the reciprocal equation or from the cubic equation.

By some means which are not evident* Ramanujan apparently discovered that

$$G^{-6}-8G^{-4}-3G^{-3}\sqrt{22}-8G^{-2}+1$$
$$\equiv (G^{-3}-3G^{-1}-\tfrac{1}{2}\sqrt{22})^2-\tfrac{1}{2}(2G^{-2}+G^{-1}\sqrt{22}+3)^2,$$

and so, since $G^{-3}-3G^{-1}-\frac{1}{2}\sqrt{22}$ is obviously negative, we have

$$G^{-3}-3G^{-1}-\tfrac{1}{2}\sqrt{22}=-(2G^{-2}+G^{-1}\sqrt{22}+3)/\sqrt{2}.$$

When we calculate the real root of this cubic in $G^{-1}$, we obtain Ramanujan's second result

$$\frac{1}{G_{121}}=\frac{1}{3\sqrt{2}}[(11-3\sqrt{11})^{\frac{1}{3}}\{(3\sqrt{11}+3\sqrt{3}-4)^{\frac{1}{3}}+(3\sqrt{11}-3\sqrt{3}-4)^{\frac{1}{3}}\}-2].$$

It is worth noting that, in a similar manner,

$$G^3-3G-\tfrac{1}{2}\sqrt{22}=+(2G^2+G\sqrt{22}+3)/\sqrt{2},$$

and hence

$$G_{121}=\frac{1}{3\sqrt{2}}[(11+3\sqrt{11})^{\frac{1}{3}}\{(3\sqrt{11}+3\sqrt{3}+4)^{\frac{1}{3}}+(3\sqrt{11}-3\sqrt{3}+4)^{\frac{1}{3}}\}+2].$$

(xviii) $\mathbf{n=169=13^2}$. (G. II, class-number 3.)

It is a fairly straightforward piece of work to derive the value of $G_{169}$ from Schläfli's† form of the modular equation of order 13; but this equation was apparently unknown to Ramanujan,‡ and he gave the modular equation of order 13 in the complementary forms

$$m=\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}},$$

$$\frac{13}{m}=\sqrt[4]{\frac{\alpha}{\beta}}+\sqrt[4]{\frac{1-\alpha}{1-\beta}}-\sqrt[4]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}-4\sqrt[6]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}},$$

where $m$ is the multiplier.

As a preliminary to the determination of $G_{169}$, I propose to show

* Once the form of the result is suspected, it is not very difficult to obtain the actual result by trial. For instance, I have found it quite feasible (in investigating $G_{361}$) to derive two quintic equations from a reciprocal decimic equation by trial, and the derivation of two cubic equations from a reciprocal sextic is, of course, much less troublesome.

† *Journal für Math.* 72 (1870), 360–9.

‡ Although Ramanujan constructed numerous equations of Schläfli's type, he left no indication that he was acquainted with Schläfli's equation of order 13.

how Schläfli's form of the equation is derivable from Ramanujan's forms. While the presumption that Ramanujan carried out this substantial piece of algebra (or its equivalent) might not be a sufficient justification for inserting it here, the transformation of Ramanujan's equation into Schläfli's equation is of considerable importance in the theory of modular equations. For Schläfli's equation has hitherto been constructed by indirect methods only, by which the general form of the equation is first determined and the numerical values of the coefficients are then found by a process of verification. On the other hand, Ramanujan's equations are obtainable by direct transformations* of $q$-series and $q$-products, so the transformation of Ramanujan's equations into Schläfli's equation forms the final stage of a direct proof of Schläfli's equation.

After this preamble, eliminate $m$ from Ramanujan's equations and write

$$Q = \sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}};$$

we thus get

$$\left\{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-Q^6-4Q^4\right\}\left\{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-1-4Q^2\right\} = 13Q^6,$$

whence

$$\begin{aligned}2\left\{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}\right\} &= 1+4Q^2+4Q^4+Q^6+\\ &\quad +\sqrt{(1+8Q^2+8Q^4+18Q^6+8Q^8+8Q^{10}+Q^{12})}\\ &= 2Q^3\Theta,\end{aligned}$$

say; we shall subsequently write $\Theta'$ for the surd conjugate to $\Theta$, so that

$$\Theta+\Theta' = Q^{-3}+4Q^{-1}+4Q+Q^3,$$
$$\Theta\Theta' = 4Q^{-2}+4+4Q^2.$$

By successive squaring it is easy to prove that

$$\frac{\beta}{\alpha}+\frac{1-\beta}{1-\alpha} = Q^{12}(\Theta^4-4\Theta^2+2).$$

Now write $P = \sqrt[24]{\{16\alpha\beta(1-\alpha)(1-\beta)\}}$,

so that $4\alpha(1-\alpha) = P^{12}Q^{-12}$, $4\beta(1-\beta) = P^{12}Q^{12}$;

when these values of $\alpha$ and $\beta$ are substituted into the last form of

* These transformations have not yet been published.

the modular equation, it becomes

$$\frac{1-\sqrt{(1-P^{12}Q^{12})}}{1-\sqrt{(1-P^{12}Q^{-12})}}+\frac{1+\sqrt{(1-P^{12}Q^{12})}}{1+\sqrt{(1-P^{12}Q^{-12})}}=Q^{12}(\Theta^4-4\Theta^2+2),$$

the signs of the newly-introduced radicals needing modifications when $\alpha>\frac{1}{2}$ and when $\beta>\frac{1}{2}$.

On clearing of fractions we get

$$2-2\sqrt{\{(1-P^{12}Q^{12})(1-P^{12}Q^{-12})\}}=P^{12}(\Theta^4-4\Theta^2+2);$$

and the result of making this equation rational in $P$ is

$$P^{12}=\frac{4\{\Theta^2-2+(Q^6+Q^{-6})\}\{\Theta^2-2-(Q^6+Q^{-6})\}}{\Theta^2(\Theta^2-4)(\Theta^2-2)^2}.$$

Now write $Q^2+Q^{-2}=v$; we find immediately that

$$\Theta^2+\Theta'^2=v^3+8v^2+13v+10,$$

$$(\Theta^2+v^3-3v-2)(\Theta'^2+v^3-3v-2)=2v(v+1)^2(v+2)(v^2-3),$$

$$(\Theta^2-v^3+3v-2)(\Theta'^2-v^3+3v-2)=-8v(v+1)^2(v^2-3),$$

$$(\Theta^2-4)(\Theta'^2-4)=-4(v+1)^2(v+2),$$

$$(\Theta^2-2)(\Theta'^2-2)=-2v(v^2-3).$$

It follows from these formulae that the result of multiplying the expression for $P^{12}$ by the conjugate expression (with $\Theta$ replaced by $\Theta'$) is unity; this indicates that substantial reductions can be made in the expression for $P^{12}$.

To effect these reductions observe that

$$(\Theta^2-v^3+3v-2)(\Theta'^2-4)=-(v+1)^2\{(v-2)\Theta'^2+16\},$$

and so

$$P^{12}=\frac{\{(v-2)\Theta'^2+16\}\{\Theta^2+v^3-3v-2\}}{(v+2)\Theta^2(\Theta^2-2)^2}$$

$$=\frac{\{(v-2)\Theta'^2+16\}\{16\Theta^2+(v-2)\Theta^2\Theta'^2\}}{16(v+2)\Theta^2(\Theta^2-2)^2}.$$

Consequently

$$\pm 4P^6\sqrt{(v+2)}=\frac{(v-2)\Theta'^2+16}{\Theta^2-2}$$

$$=\frac{(v-2)(v^3+8v^2+10v+13)+16-(v-2)\Theta^2}{\Theta^2-2}$$

$$=\frac{v^4+6v^3-3v^2-18v}{\Theta^2-2}-v+2$$

$$=-\tfrac{1}{2}(v+6)(\Theta'^2-2)-v+2$$

$$=8-\tfrac{1}{2}(v+6)\Theta'^2,$$

and therefore

$$\pm P^6 = \tfrac{1}{16}[(v+3)(v+6)\sqrt{(v^3+8v^2+5v+2)} - \\ -(v^3+12v^2+37v+14)\sqrt{(v+2)}].$$

Taking the reciprocal of each side, we have

$$\pm P^{-6} = \tfrac{1}{16}[(v+3)(v+6)\sqrt{(v^3+8v^2+5v+2)} + \\ +(v^3+12v^2+37v+14)\sqrt{(v+2)}],$$

and, since the expression on the right of the last equation is positive, the upper signs in the ambiguities must be taken.

Replacing $v$ by $Q^{-2}+Q^2$, we obtain the modular equation of order 13 in the forms

$$P^{-6}+P^6 \\ = \frac{1}{8}\left(\frac{1}{Q^2}+3+Q^2\right)\left(\frac{1}{Q^2}+6+Q^2\right)\sqrt{\left(\frac{1}{Q^6}+\frac{8}{Q^4}+\frac{8}{Q^2}+18+8Q^2+8Q^4+Q^6\right)},$$

$$P^{-6}-P^6 = \frac{1}{8}\left(\frac{1}{Q}+Q\right)\left(\frac{1}{Q^6}+\frac{12}{Q^4}+\frac{40}{Q^2}+38+40Q^2+12Q^4+Q^6\right),$$

where

$$P = \sqrt[24]{\{16\alpha\beta(1-\alpha)(1-\beta)\}}, \qquad Q = \sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$$

The second of these forms is Schläfli's equation.

We now determine the value of $G_{169}$. Take $\alpha = \frac{1}{2}$, and then

$$P^{-1} = Q^{-1} = G_{169}.$$

Hence, omitting the suffix 169 for brevity, we have

$$G^6-\frac{1}{G^6} = \frac{1}{8}\left(G+\frac{1}{G}\right)\left(G^6+12G^4+40G^2+38+\frac{40}{G^2}+\frac{12}{G^4}+\frac{1}{G^6}\right).$$

Discard the irrelevant factor $G+G^{-1}$, write $v$ in place of $G^2+G^{-2}$, and square the resulting equation; we get

$$(v^3+12v^2+37v+14)^2-64(v-2)(v^2-1)^2 = 0,$$

that is to say,

$$(v^3-20v^2-27v-18)^2 = 0,$$

so that

$$(G+G^{-1})^6-26(G+G^{-1})^4+65(G+G^{-1})^2-52 = 0.$$

Therefore

$$\{(G+G^{-1})^3+13(G+G^{-1})\}^2 = 52\{(G+G^{-1})^2+1\}^2,$$

that is to say

$$(G+G^{-1})^3-2(G+G^{-1})^2\sqrt{13}+13(G+G^{-1}) = \sqrt{52},$$

so that

$$(G+G^{-1})^{\frac{3}{2}}-(G+G^{-1})^{\frac{1}{2}}\sqrt{13} = +\sqrt[4]{52},$$

with the correct sign because, by direct computation of the $q$-product,

$$G_{169} = 4{\cdot}61083\,6185, \qquad (G_{169}+G_{169}^{-1})^{\frac{1}{2}} = 2{\cdot}19720\,6541.$$

By solving this cubic equation, we obtain Ramanujan's first result

$$G_{169}+\frac{1}{G_{169}} = \left(\frac{13}{4}\right)^{\frac{1}{6}}\left\{\left(1+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(1-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\}^2.$$

We return to the equation

$$(G+G^{-1})^3-2(G+G^{-1})^2\sqrt{13}+13(G+G^{-1})-2\sqrt{13} = 0.$$

By some means which are not evident Ramanujan apparently discovered that

$$G^{-6}-2G^{-5}\sqrt{13}+16G^{-4}-6G^{-3}\sqrt{13}+16G^{-2}-2G^{-1}\sqrt{13}+1$$
$$\equiv (G^{-3}-G^{-2}\sqrt{13}+\tfrac{7}{2}G^{-1}-\tfrac{1}{2}\sqrt{13})^2-(2G^{-2}-\tfrac{1}{2}\sqrt{13}\,G^{-1}+\tfrac{3}{2})^2,$$

and so, with the correct sign,

$$G^{-3}-G^{-2}\sqrt{13}+\tfrac{7}{2}G^{-1}-\tfrac{1}{2}\sqrt{13} = -(2G^{-2}-\tfrac{1}{2}\sqrt{13}+\tfrac{3}{2}).$$

By solving this cubic we obtain Ramanujan's second result

$$\frac{1}{G_{169}} = \frac{1}{3}\Big[(\sqrt{13}-2)+$$
$$+\left(\frac{13-3\sqrt{13}}{2}\right)^{\frac{1}{3}}\left\{\left(3\sqrt{3}-\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}-\left(3\sqrt{3}+\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}\right\}\Big].$$

It is worth noting that, in a similar manner,

$$G^3-(2+\sqrt{13})G^2+\frac{7+\sqrt{13}}{2}G-\frac{3+\sqrt{13}}{2} = 0,$$

and hence

$$G_{169} = \frac{1}{3}\Big[(\sqrt{13}+2)+$$
$$+\left(\frac{13+3\sqrt{13}}{2}\right)^{\frac{1}{3}}\left\{\left(\frac{11+\sqrt{13}}{2}+3\sqrt{3}\right)^{\frac{1}{3}}+\left(\frac{11+\sqrt{13}}{2}-3\sqrt{3}\right)^{\frac{1}{3}}\right\}\Big].$$

(xix) $\mathbf{n = 225 = 3^2\times 5^2}$. (G. IV, class-number 2.)

One of the theorems* on modular equations discovered by Ramanujan is

*If $\alpha$, $\beta$, $\gamma$, $\delta$ be of the first, third, fifth, and fifteenth degrees, and if*

$$P = \sqrt[48]{\{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}}$$

* The version given in the *Collected Papers* (1927), pp. xxix and 353, differs slightly in notation.

*and*
$$Q = \sqrt[16]{\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}},$$

*then*
$$Q+\frac{1}{Q} = \sqrt{2}\left(P+\frac{1}{P}\right).$$

Take $\alpha = \frac{1}{2}$ and then the result of squaring this modular equation is

$$\frac{G_9^3 G_{25}^3}{G_1^3 G_{225}^3}+\frac{G_1^3 G_{225}^3}{G_9^3 G_{25}^3} = 2\left(\frac{1}{G_1 G_9 G_{25} G_{225}}+1+G_1 G_9 G_{25} G_{225}\right).$$

Again, take $\beta = \frac{1}{2}$ and observe that $G_{1/9} = G_9$; the result of squaring the modular equation then is

$$\frac{G_1^3 G_{25/9}^3}{G_9^3 G_{25}^3}+\frac{G_9^3 G_{25}^3}{G_1^3 G_{25/9}^3} = 2\left(\frac{1}{G_9 G_1 G_{25/9} G_{25}}+1+G_9 G_1 G_{25/9} G_{225}\right).$$

Hence two of the roots of the equation

$$\frac{G_9^3 G_{25}^3}{G_1^3 x^3}+\frac{G_1^3 x^3}{G_9^3 G_{25}^3} = 2\left(\frac{1}{G_1 G_9 G_{25} x}+1+G_1 G_9 G_{25} x\right)$$

are
$$x = G_{225}, \qquad x = G_{25/9};$$

and, by Descartes's rule of signs, the equation has not more than two positive roots, so these are the only positive roots. We now proceed to solve the equation.

It is known* that

$$G_1 = 1, \qquad G_9 = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{\frac{1}{3}}, \qquad G_{25} = \frac{\sqrt{5}+1}{2};$$

and hence, if we write
$$x = G_9^2 G_{25} X,$$

the equation in $X$ does not contain a cubic irrationality. This equation is

$$G_9^3 X^3+G_9^{-3}X^{-3} = 2(G_9^3 G_{25}^2 X+G_9^{-3}G_{25}^{-2}X^{-1}+1),$$

that is to say,

$$(G_9^3+G_9^{-3})(X^3+X^{-3})+(G_9^3-G_9^{-3})(X^3-X^{-3})$$
$$= 2(G_9^3 G_{25}^2+G_9^{-3}G_{25}^{-2})(X+X^{-1})+4+2(G_9^3 G_{25}^2-G_9^{-3}G_{25}^{-2})(X-X^{-1}),$$

which reduces to

$$\{(X^3+X^{-3})\sqrt{6}-(3\sqrt{6}+\sqrt{10})(X+X^{-1})-4\}+$$
$$+(X-X^{-1})\{(X^2+1+X^{-2})\sqrt{2}-(3\sqrt{2}+\sqrt{30})\} = 0.$$

* See, for example, the list in Weber's *Algebra*. The results quoted were, of course, known to Ramanujan.

Now

$$(X^2+1+X^{-2})\sqrt{2}-(3\sqrt{2}+\sqrt{30})$$
$$=\left(X+X^{-1}-\frac{\sqrt{10}+\sqrt{6}}{2}\right)\left(X+X^{-1}+\frac{\sqrt{10}+\sqrt{6}}{2}\right)\sqrt{2}$$

and

$$(X^3+X^{-3})\sqrt{6}-(3\sqrt{6}+\sqrt{10})(X+X^{-1})-4=\left(X+X^{-1}-\frac{\sqrt{10}+\sqrt{6}}{2}\right)\times$$
$$\times\{(X+X^{-1})^2\sqrt{6}+(X+X^{-1})(3+\sqrt{15})+2(\sqrt{10}-\sqrt{6})\},$$

so that the positive roots of the sextic in $X$ are the roots of the quadratic

$$X+X^{-1}=\frac{\sqrt{10}+\sqrt{6}}{2},$$

the larger root obviously appertaining to $G_{225}$ and the smaller to $G_{25/9}$.

The larger root is $$\frac{(\sqrt{5}+\sqrt{3})+\sqrt[4]{60}}{2\sqrt{2}},$$

and therefore

$$G_{225}=(2+\sqrt{3})^{\frac{1}{4}}\left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{(\sqrt{5}+\sqrt{3})+\sqrt[4]{60}}{2\sqrt{2}}\right).$$

(xx) $\mathbf{n=325=5^2\times13}$. (G. IV, class-number 3.)

It is evident that $G_{325}$ and $G_{13}$ are connected by the modular equation of order 5; and from Schläfli's equation of this order we have

$$\left(\frac{G_{325}}{G_{13}}\right)^3+\left(\frac{G_{13}}{G_{325}}\right)^3=2\left\{G_{13}^2G_{325}^2-\frac{1}{G_{13}^2G_{325}^2}\right\}.$$

Now* $$G_{13}=\left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{4}},$$

and so, if we put $$G_{325}=\left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{4}}t,$$

then $t$ is a root of the equation

$$t^3+t^{-3}=(\sqrt{13}+3)t^2-(\sqrt{13}-3)t^{-2}.$$

It is easier to verify than to discover that

$$4t^6-(12+4\sqrt{13})t^5-(12-4\sqrt{13})t+4$$
$$\equiv 5[t^3-\tfrac{1}{2}t^2+\tfrac{1}{2}t-1-\tfrac{1}{2}(t^2+t)\sqrt{13}]^2-[t^3+\tfrac{7}{2}t^2+\tfrac{7}{2}t+1-\tfrac{1}{2}(t^2-t)\sqrt{13}]^2.$$

* See, for example, the list in Weber's *Algebra*.

From the $q$-product it follows that

$$G_{325} = 8{\cdot}90435\,9573, \qquad t = 6{\cdot}60515\,3594,$$

and so, selecting the correct sign, we obtain Ramanujan's result that, if

$$G_{325} = \left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{4}} t,$$

then

$$t^3+t^2\left(\frac{\sqrt{13}-1}{2}\right)^2+t\left(\frac{\sqrt{13}+1}{2}\right)^2+1 = +\sqrt{5}.\left\{t^3-t^2\frac{\sqrt{13}+1}{2}-t\frac{\sqrt{13}-1}{2}-1\right\}.$$

(xxi) $\mathbf{n = 363 = 3\times 11^2}$. (G. II, class-number 6.)

We again work with the modular equation of order 11 which was examined in the investigation of $n = 121$. From the equations

$$2\alpha\beta = A-BR, \qquad 2(1-\alpha)(1-\beta) = A+BR,$$

it follows that $\alpha+\beta = 1-BR,$

and consequently $\alpha$ and $\beta$ are the roots of the equation

$$X^2-(1-BR)X+\tfrac{1}{2}(A-BR) = 0,$$

so that $(X^2-X+\tfrac{1}{2}A)^2 = B^2R^2(X-\tfrac{1}{2})^2.$

Hence $\alpha(1-\alpha)$ and $\beta(1-\beta)$ are the roots of the equation

$$(X-\tfrac{1}{2}A)^2 = (A^2-\tfrac{1}{4}t^{12})(\tfrac{1}{4}-X),$$

i.e. $X^2-pX+\tfrac{1}{16}t^{12} = 0,$

where $p = A-A^2+\tfrac{1}{4}t^{12}$

$$= 2t(1-4t+4t^2-t^3)^2(2-3t+2t^2)^2-\tfrac{1}{2}t^6.$$

Now take $\sqrt{\alpha}$ and $\sqrt{\beta}$ to be the moduli associated with the parameters $e^{-\pi\sqrt{3}}$ and $e^{-\pi\sqrt{363}}$, so that

$$4\alpha(1-\alpha) = G_3^{-24}, \qquad 4\beta(1-\beta) = G_{363}^{-24}, \qquad G_3\,G_{363} = t^{-\frac{1}{2}}.$$

Then $\frac{1}{4}G_3^{-24}$ is a root of the equation

$$X^2-pX+\tfrac{1}{16}t^{12} = 0$$

where $t$ has the value just specified and $p$ is the function of $t$ given above.

It is well known that $G_3 = 2^{\frac{1}{4}}$, and so $G_{363}$ is the value of $2^{-\frac{1}{12}}t^{-\frac{1}{2}}$ where $t$ is one of the roots of the equation

$$\tfrac{1}{256}-\tfrac{1}{16}\{2t(1-4t+4t^2-t^3)^2(2-3t+2t^2)^2-\tfrac{1}{2}t^6\}+\tfrac{1}{16}t^{12} = 0.$$

Hence, if $x = 2^{-\frac{1}{24}}G_{363}$, then $2t = 1/x^2$, and $x$ is one of the roots of the equation

$$\left(\frac{1}{64x^{12}}+\frac{1}{4}\right)^2 = \frac{1}{x^2}\left(1-\frac{2}{x^2}+\frac{1}{x^4}-\frac{1}{8x^6}\right)^2\left(2-\frac{3}{2x^2}+\frac{1}{2x^4}\right)^2.$$

From the $q$-product it follows that

$$G_{363} = 10{\cdot}18267\,076, \qquad x = 7{\cdot}62838\,4185;$$

and so we deduce from the last equation that, with the correct sign,*

$$16x^{12}+1 = +4x(8x^6-16x^4+8x^2-1)(4x^4-3x^2+1).$$

This equation may be written in the form

$$(4x^6-16x^5+34x^4-18x^3-2x^2+2x+1)^2-132(2x^5-3x^4+x^3)^2 = 0,$$

and so, again with the correct sign,

$$4x^6-16x^5+34x^4-18x^3-2x^2+2x+1-2(2x^5-3x^4+x^3)\sqrt{33} = 0.$$

This equation may be written in the form

$$\{2x^3-x^2(4+\sqrt{33})-x-1\}^2-(11+2\sqrt{33})(x^4+2x^3+x^2) = 0,$$

and hence, taking the correct sign, we obtain Ramanujan's result that $G_{363} = 2^{\frac{1}{24}}x$ where

$$2x^3-x^2\{4+\sqrt{33}+\sqrt{(11+2\sqrt{33})}\}-x\{1+\sqrt{(11+2\sqrt{33})}\}-1 = 0.$$

(xxii) $\mathbf{n = 1225} = 5^2\times 7^2$. (G. IV, class-number 4.)

To obtain an equation of degree not exceeding 4 which is satisfied by $G_{1225}$, we have recourse to the following theorem:†

*If $\alpha, \beta, \gamma, \delta$ be of the first, fifth, seventh, and thirty-fifth degrees, and if*

$$P = \sqrt[24]{\{16\alpha\delta(1-\alpha)(1-\delta)\}}, \qquad Q = \sqrt[24]{\{16\beta\gamma(1-\beta)(1-\gamma)\}},$$

*then*

$$\frac{Q-P^3}{Q+Q^3} = \frac{P+P^3}{Q^3-P}.$$

By putting $\alpha = \frac{1}{2}$ in this result, we have

$$\frac{G_{25}^{-1}G_{49}^{-1}-G_1^{-3}G_{1225}^{-3}}{G_{25}^{-1}G_{49}^{-1}+G_{25}^{-3}G_{49}^{-3}} = \frac{G_1^{-1}G_{1225}^{-1}+G_1^{-3}G_{1225}^{-3}}{G_{25}^{-3}G_{49}^{-3}-G_1^{-1}G_{1225}^{-1}},$$

and again, by putting $\beta = \frac{1}{2}$, we have

$$\frac{G_1^{-1}G_{49/25}^{-1}-G_{1/25}^{-3}G_{49}^{-3}}{G_1^{-1}G_{49/25}^{-1}+G_1^{-3}G_{49/25}^{-3}} = \frac{G_{1/25}^{-1}G_{49}^{-1}+G_{1/25}^{-3}G_{49}^{-3}}{G_1^{-3}G_{49/25}^{-3}-G_{1/25}^{-1}G_{49}^{-1}}.$$

* An equation equivalent to this has been given by Mitra, *Bulletin Calcutta Math. Soc.* 19 (1928), 150.

† This is one of the numerous modular equations which are not to be found anywhere except in Ramanujan's note-book.

Since $G_1 = 1$ and $G_{1/25} = G_{25}$, these results respectively show that

$$x = \frac{G_1 G_{1225}}{G_{25} G_{49}} \quad \text{and} \quad x = \frac{G_1 G_{49/25}}{G_{25} G_{49}}$$

are two roots of the quartic

$$x^4 - x^3(1 + 2G_{25}^2 G_{49}^2) - x(1 + 2G_{25}^{-2} G_{49}^{-2}) + 1 = 0;$$

and, by Descartes's rule of signs, they are the only positive roots.

The values of $G_{25}$ and $G_{49}$ which occur in the coefficients of the quartic are as follows:*

$$G_{25} = \frac{\sqrt{5}+1}{2}, \qquad G_{49} = \frac{\sqrt{(4+\sqrt{7})}+\sqrt[4]{7}}{2}.$$

The procedure of solving the quartic algebraically does not look promising when the reducing cubic is written down; we evade the difficulties of the direct solution in the following manner.

It is evident that $G_{49}$ and $G_{1225}$ are connected by the modular equation of order 5, so that

$$\frac{G_{1225}^3}{G_{49}^3} + \frac{G_{49}^3}{G_{1225}^3} = 2(G_{49}^2 G_{1225}^2 - G_{49}^{-2} G_{1225}^{-2});$$

also $G_{49/25}$ and $G_{49}$ are connected by the modular equation of order 5, so that†

$$\frac{G_{49}^3}{G_{49/25}^3} + \frac{G_{49/25}^3}{G_{49}^3} = 2(G_{49/25}^2 G_{49}^2 - G_{49/25}^{-2} G_{49}^{-2}).$$

These results respectively show that

$$x = \frac{G_1 G_{1225}}{G_{25} G_{49}} \quad \text{and} \quad x = \frac{G_1 G_{49/25}}{G_{25} G_{49}}$$

are two roots of the sextic

$$G_{25}^3 x^6 - 2G_{25}^2 G_{49}^4 x^5 + 2G_{25}^{-2} G_{49}^{-4} x + G_{25}^{-3} = 0.$$

Hence the equations

$$x^4 - x^3(1 + 2G_{25}^2 G_{49}^2) - x(1 + 2G_{25}^{-2} G_{49}^{-2}) + 1 = 0,$$

$$G_{25}^3 x^6 - 2G_{25}^2 G_{49}^4 x^5 + 2G_{25}^{-2} G_{49}^{-4} x + G_{25}^{-3} = 0$$

have two roots in common, namely,

$$\frac{G_1 G_{1225}}{G_{25} G_{49}}, \qquad \frac{G_1 G_{49/25}}{G_{25} G_{49}},$$

and, when their product has been determined, it proves to be an

* See, for example, the list in Weber's *Algebra*.

† These are both modular equations of Schläfli's type.

easy matter to determine each of them separately with the help of the quartic equation. This product will be denoted by $\lambda^2$, where $\lambda$ is chosen to be positive.

Now consider the general problem in which it is given that the equations

$$a_0x^4+a_1x^3+a_3x+a_4 = 0,$$
$$b_0x^6+b_1x^5+b_5x+b_6 = 0$$

have two roots in common and it is required to find the product of these two roots.

The common roots satisfy the equation

$$b_0x^6+b_1x^5+b_5x+b_6-(a_0x^4+a_1x^3+a_3x+a_4)\left(\frac{b_0}{a_0}x^2+\theta x+\frac{b_6}{a_4}\right) = 0;$$

and this equation reduces to a cubic when $\theta$ is chosen so that

$$a_0^2\theta = a_0b_1-a_1b_0.$$

It also reduces to a cubic when $\theta$ is replaced by $\theta'$ where

$$a_4^2\theta' = a_4b_5-a_3b_6.$$

Let these cubics respectively be taken to be

$$A_0x^3+A_1x^2+A_2x+A_3 = 0,$$
$$B_0x^3+B_1x^2+B_2x+B_3 = 0.$$

From these two cubics we obtain two (identical) quadratics

$$(A_0B_3-A_3B_0)x^2+(A_1B_3-A_3B_1)x+(A_2B_3-A_3B_2) = 0,$$
$$(A_0B_1-A_1B_0)x^2+(A_0B_2-A_2B_0)x+(A_0B_3-A_3B_0) = 0,$$

which (if they are not nugatory*) have the common roots of the quartic and sextic as their roots. Hence

$$\lambda^2 = \frac{A_2B_3-A_3B_2}{A_0B_3-A_3B_0} = \frac{A_0B_3-A_3B_0}{A_0B_1-A_1B_0},$$

so that

$$\lambda^4 = \frac{A_2B_3-A_3B_2}{A_0B_1-A_1B_0}.$$

Now $a_0^2a_4^2(A_2B_3-A_3B_2)$

$$= \begin{vmatrix} a_4(\theta a_0a_3+b_0a_4), & b_0a_3a_4+b_6a_0a_1 \\ a_0(\theta a_4^2+b_6a_3-b_5a_4), & a_4(\theta' a_0a_3+b_0a_4) \end{vmatrix}.$$

When this equation is simplified by taking $a_0 = a_4 = 1$ and $b_0b_6 = 1$, it becomes

$$A_2B_3-A_3B_2$$
$$= (\theta\theta'-1)a_3^2+b_0^2-b_6^2a_1a_3+\theta' b_0a_3-\theta b_6a_1+b_5(b_0a_3+b_6a_1);$$

* They are nugatory if the quartic and sextic have more than two roots in common.

and, similarly, we have

$$A_0B_1-A_1B_0 = (\theta\theta'-1)a_1^2+b_6^2-b_0^2a_1a_3+\theta b_6a_1-\theta'b_0a_3+b_1(b_6a_1+b_0a_3).$$

Hence we may write

$$A_2B_3-A_3B_2 = U+V, \qquad A_0B_1-A_1B_0 = U-V,$$

where

$$U = \tfrac{1}{2}(a_1^2+a_3^2)(\theta\theta'-1)-\tfrac{1}{2}(b_0^2+b_6^2)(a_1a_3-1)+\tfrac{1}{2}(b_1+b_5)(b_0a_3+b_6a_1),$$
$$V = \tfrac{1}{2}(a_1^2-a_3^2)(1-\theta\theta')+\tfrac{1}{2}(b_0^2-b_6^2)(a_1a_3+1)+\theta'b_0a_3-\theta b_6a_1+$$
$$+\tfrac{1}{2}(b_5-b_1)(b_0a_3+b_6a_1).$$

Now substitute for $a_1$, $a_3$, $b_0$, $b_1$, $b_5$, $b_6$, $\theta$, $\theta'$ their values in the form of surds for the special quartic and sextic which we have to consider. It is readily found that

$$\tfrac{1}{2}(a_1^2+a_3^2)(\theta\theta'-1) = 385+121\sqrt{7}-(31\sqrt{70}+112\sqrt{10})\sqrt[4]{7},$$
$$\tfrac{1}{2}(b_0^2+b_6^2)(a_1a_3-1) = \tfrac{1}{2}[180+54\sqrt{7}+(9\sqrt{70}+9\sqrt{10})\sqrt[4]{7}],$$
$$\tfrac{1}{2}(b_1+b_5)(b_0a_3+b_6a_1) = \tfrac{1}{2}[194+80\sqrt{7}+(25\sqrt{70}+67\sqrt{10})\sqrt[4]{7}],$$

so that
$$U = 392+134\sqrt{7}-(23\sqrt{70}+83\sqrt{10})\sqrt[4]{7};$$

also
$$\tfrac{1}{2}(a_1^2-a_3^2)(1-\theta\theta') = 130\sqrt{5}+62\sqrt{35}-(42\sqrt{14}+48\sqrt{2})\sqrt[4]{7},$$
$$\tfrac{1}{2}(b_0^2-b_6^2)(a_1a_3-1) = 48\sqrt{5}+12\sqrt{35}-(10\sqrt{14}+10\sqrt{2})\sqrt[4]{7},$$
$$\theta'b_0a_3-\theta b_6a_1 = -\tfrac{1}{2}[78\sqrt{5}+20\sqrt{35}-(23\sqrt{14}+65\sqrt{2})\sqrt[4]{7}],$$
$$\tfrac{1}{2}(b_5-b_1)(b_0a_3+b_6a_1) = -\tfrac{1}{2}[150\sqrt{5}+60\sqrt{35}+(33\sqrt{14}+87\sqrt{2})\sqrt[4]{7}],$$

so that
$$V = 64\sqrt{5}+34\sqrt{35}-(37\sqrt{14}+49\sqrt{2})\sqrt[4]{7}.$$

Now
$$(U^2-V^2)\lambda^4 = (U+V)^2,$$

and, since*

$$U^2-V^2 = 4\,34912+1\,65248\sqrt{7}-(32208\sqrt{70}+84336\sqrt{10})\sqrt[4]{7}$$
$$= [312+132\sqrt{7}-(22\sqrt{70}+70\sqrt{10})\sqrt[4]{7}]^2,$$

we have

$$[312+132\sqrt{7}-(22\sqrt{70}+70\sqrt{10})\sqrt[4]{7}]^2\lambda^4$$
$$= [392+134\sqrt{7}+64\sqrt{5}+34\sqrt{35}-(23\sqrt{70}+83\sqrt{10}+37\sqrt{14}+49\sqrt{2})\sqrt[4]{7}]^2.$$

Remove the irrationality $\sqrt[4]{7}$ from the left by multiplying through by the conjugate surd; we get

$$[(312+132\sqrt{7})^2-(22\sqrt{70}+70\sqrt{10})^2\sqrt{7}]^2\lambda^4$$
$$= [5600+32\sqrt{7}+32\sqrt{5}+800\sqrt{35}-$$
$$-(128\sqrt{70}-928\sqrt{10}-928\sqrt{14}+896\sqrt{2})\sqrt[4]{7}]^2,$$

* The work of expressing the surd form of $U^2-V^2$ as a perfect square is rather tedious.

that is to say,

$$[116-16\sqrt{7}]^2\lambda^4 = [175+\sqrt{7}+\sqrt{5}+25\sqrt{35}+(29-4\sqrt{7})(\sqrt{14}+\sqrt{10})\sqrt[4]{7}]^2.$$

Divide through by $[29-4\sqrt{7}]^2$; we get

$$16\lambda^4 = [7+\sqrt{7}+\sqrt{5}+\sqrt{35}+(\sqrt{14}+\sqrt{10})\sqrt[4]{7}]^2,$$

and so, since $\lambda^2$ is positive, we have

$$\lambda^2 = \frac{\sqrt{14}+\sqrt{10}}{2}\,\frac{\sqrt{14}+\sqrt{2}+2\sqrt[4]{7}}{4}$$
$$= G_{49}\sqrt{(6+\sqrt{35})}.$$

Now that $\lambda$ has been obtained, it is an easy matter to complete the work. The quartic, since the product of all of its roots is unity, may obviously be written in the form

$$(\lambda^{-1}x^2-2Ax+\lambda)(\lambda x^2-2Bx+\lambda^{-1}) = 0,$$

the first factor giving the positive roots; by multiplying out and equating coefficients, we have

$$A\lambda+B\lambda^{-1} = \tfrac{1}{2}+G_{25}^2\,G_{49}^2,$$
$$A\lambda^{-1}+B\lambda = \tfrac{1}{2}+G_{25}^{-2}G_{49}^{-2},$$
$$4AB+\lambda^2+\lambda^{-2} = 0.$$

The result of eliminating $A$ and $B$ from these three equations is

$$(\lambda^2+\lambda^{-2})^3+(1+2G_{25}^2\,G_{49}^2+2G_{25}^{-2}G_{49}^{-2})(\lambda^2+\lambda^{-2})$$
$$= (1+2G_{25}^2\,G_{49}^2)^2+(1+2G_{25}^{-2}G_{49}^{-2})^2,$$

which is, in effect, the reducing cubic of the quartic; the reader may find it interesting to verify that it is satisfied by the value which has just been obtained for $\lambda^2$.

Again, by eliminating $B$ from the first two of the three equations, we have

$$2A(\lambda^2-\lambda^{-2}) = (\lambda-\lambda^{-1})+2(\lambda G_{25}^2\,G_{49}^2-\lambda^{-1}G_{25}^{-2}G_{49}^{-2}),$$

so that $A$ is positive and

$$4A^2(\lambda^2-\lambda^{-2})^2 = \lambda^2(1+2G_{25}^2\,G_{49}^2)^2+\lambda^{-2}(1+2G_{25}^{-2}G_{49}^{-2})^2-$$
$$-2(1+2G_{25}^2\,G_{49}^2)(1+2G_{25}^{-2}G_{49}^{-2}),$$

whence it follows that

$$[40+24\sqrt{7}+2(\sqrt{70}+7\sqrt{10})\sqrt[4]{7}]A^2$$
$$= \tfrac{1}{2}[2075+793\sqrt{7}+(152\sqrt{70}+409\sqrt{10})\sqrt[4]{7}].$$

Remove the irrationality $\sqrt[4]{7}$ from the left by multiplying through by the conjugate surd; we get

$$16(107-20\sqrt{7})A^2 = 5002+1490\sqrt{7}+(322\sqrt{70}+872\sqrt{10})\sqrt[4]{7},$$

and then, by multiplying through by $107+20\sqrt{7}$, we have

$$1\,38384A^2 = 7\,43814+2\,59470\sqrt{7}+(51894\sqrt{70}+1\,38384\sqrt{10})\sqrt[4]{7},$$

so that
$$A = \sqrt{\left(\frac{43+15\sqrt{7}+(3\sqrt{70}+8\sqrt{10})\sqrt[4]{7}}{8}\right)}.$$

It is evident that the larger positive root of the quartic is associated with $G_{1225}$, and therefore

$$G_{1225} = G_{25}\,G_{49}\lambda\{A+\sqrt{(A^2-1)}\},$$

that is to say

$$G_{1225} = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{\sqrt{(4+\sqrt{7})}+\sqrt[4]{7}}{2}\right)^{\frac{3}{2}}\sqrt{\left(\frac{\sqrt{14}+\sqrt{10}}{2}\right)}\times$$
$$\times\left[\sqrt{\left(\frac{43+15\sqrt{7}+(3\sqrt{70}+8\sqrt{10})\sqrt[4]{7}}{8}\right)}+\right.$$
$$\left.+\sqrt{\left(\frac{35+15\sqrt{7}+(3\sqrt{70}+8\sqrt{10})\sqrt[4]{7}}{8}\right)}\right];$$

and this is Ramanujan's result.

**Singular moduli of the type $g_{2m}$**

Before discussing the special cases of $g_{62}$, $g_{94}$, and $g_{158}$, let $\sqrt{\alpha}$ and $\sqrt{\beta}$ be the moduli associated with $g_{2/m}$ and $g_{2m}$, so that $\alpha$ and $\beta$ are connected by the modular equation of order $m$, while

$$\frac{1-\beta}{2\sqrt{\beta}} = g_{2m}^{12}, \qquad \frac{1-\alpha}{2\sqrt{\alpha}} = g_{2/m}^{12} = \frac{1}{g_{2m}^{12}}.$$

Hence
$$\sqrt{\beta} = \sqrt{(1+g_{2m}^{24})}-g_{2m}^{12}, \qquad \sqrt{\alpha} = \frac{\sqrt{(1+g_{2m}^{24})}-1}{g_{2m}^{12}}.$$

Consequently, if we write

$$\alpha\beta = X^{16},$$

we have
$$(1-\alpha)(1-\beta) = 4X^8,$$

and also
$$X^{\pm 8} = \frac{\{\sqrt{(1+g_{2m}^{24})}\pm 1\}\{\sqrt{(1+g_{2m}^{24})}\mp g_{2m}^{12}\}}{g_{2m}^{12}}$$

so that
$$X^8+\frac{1}{X^8} = 2\left(g_{2m}^{12}+1+\frac{1}{g_{2m}^{12}}\right),$$

and therefore
$$(X^4+X^{-4})^2 = 2(g_{2m}^6+g_{2m}^{-6})^2.$$

In the sequel we shall obtain a set of equations of the form

$$X^4+X^{-4}+a = b(X^2+X^{-2}),$$

where $a$ and $b$ are given surds; by squaring this equation we get

$$(X^4+X^{-4})^2+(2a-b^2)(X^4+X^{-4})+a^2-2b^2=0;$$

the corresponding equation in $g_{2m}$ is

$$(g_{2m}^6+g_{2m}^{-6})^2+(2a-b^2)(g_{2m}^6+g_{2m}^{-6})/\sqrt{2}+\tfrac{1}{2}a^2-b^2=0,$$

whence $$2(g_{2m}^6+g_{2m}^{-6})=(b^2-2a)/\sqrt{2}\pm b\sqrt{(\tfrac{1}{2}b^2-2a+4)}.$$

We are now in a position to examine the specified singular moduli.

(xxiii) $\mathbf{n=62}$. (G. II, class-number 4.)

A form of the modular equation of order 31 discovered by Ramanujan is

$$1+\sqrt[4]{\{\alpha\beta\}}+\sqrt[4]{\{(1-\alpha)(1-\beta)\}}-\sqrt{\left(\frac{1+\sqrt{\{\alpha\beta\}}+\sqrt{\{(1-\alpha)(1-\beta)\}}}{2}\right)}$$
$$=\sqrt[8]{\{\alpha\beta\}}+\sqrt[8]{\{(1-\alpha)(1-\beta)\}}+\sqrt[8]{\{\alpha\beta(1-\alpha)(1-\beta)\}}.$$

Making the standard substitutions

$$\alpha\beta=X^{16},\qquad (1-\alpha)(1-\beta)=4X^8,$$

as explained above, we get

$$1+X^4+X^2\sqrt{2}-\sqrt{\left(\frac{1+X^8+2X^4}{2}\right)}=X^2+2^{\frac{1}{2}}(X+X^3).$$

By rearranging and squaring we get in succession

$$X^2+X^{-2}+\sqrt{2}=2^{\frac{1}{2}}(\sqrt{2}+1)(X+X^{-1}),$$

$$X^4+X^{-4}+4+2\sqrt{2}.(X^2+X^{-2})=(8+6\sqrt{2})(X^2+X^{-2}+2),$$

so that $$X^4+X^{-4}-12-12\sqrt{2}=(8+4\sqrt{2})(X^2+X^{-2}).$$

The standard transformation now gives

$$g_{62}^6+g_{62}^{-6}=44+30\sqrt{2}\pm 4\sqrt{(226+160\sqrt{2})};$$

the upper sign is to be taken because the lower sign makes the expression on the right less than 2. Hence

$$g_{62}^3+g_{62}^{-3}=\sqrt{(28+20\sqrt{2})}+\sqrt{(18+10\sqrt{2})}.$$

When we solve this equation regarded as a cubic in $g_{62}+g_{62}^{-1}$ by trial, we obtain Ramanujan's result

$$g_{62}+\frac{1}{g_{62}}=\frac{\sqrt{(9+5\sqrt{2})}+\sqrt{(\sqrt{2}+1)}}{2}.$$

A slight modification of this result is

$$g_{62}+g_{62}^{-1}=\tfrac{1}{2}\{1+\sqrt{(4\sqrt{2}+1)}\}\sqrt{(\sqrt{2}+1)}.$$

(xxiv) $\mathbf{n = 94}$. (G. II, class-number 4.)

A form of the modular equation of order 47 discovered by Ramanujan is

$$2\sqrt{\left(\frac{1+\sqrt{\{\alpha\beta\}}+\sqrt{\{(1-\alpha)(1-\beta)\}}}{2}\right)} = 1+\sqrt[4]{\{\alpha\beta\}}+\sqrt[4]{\{(1-\alpha)(1-\beta)\}}+$$
$$+\sqrt[3]{4}.\sqrt[24]{\{\alpha\beta(1-\alpha)(1-\beta)\}}.[1+\sqrt[8]{\{\alpha\beta\}}+\sqrt[8]{\{(1-\alpha)(1-\beta)\}}].$$

Making the standard substitutions

$$\alpha\beta = X^{16}, \qquad (1-\alpha)(1-\beta) = 4X^8,$$

as previously explained, we get

$$\sqrt{(2+2X^8+4X^4)} = 1+X^4+X^2\sqrt{2}+2^{\frac{1}{2}}X(1+X^2+2^{\frac{1}{2}}X),$$

so that $$X^2+X^{-2}-(4+3\sqrt{2}) = 2^{\frac{1}{2}}(\sqrt{2}+1)(X+X^{-1}),$$

and hence, by squaring,

$$X^4+X^{-4}+(20+12\sqrt{2}) = (16+12\sqrt{2})(X^2+X^{-2}).$$

The standard transformation now gives

$$g_{94}^6+g_{94}^{-6} = 180+126\sqrt{2}\pm(16+12\sqrt{2})\sqrt{(59+42\sqrt{2})},$$

and the upper sign is to be taken. Hence

$$g_{94}^3+g_{94}^{-3} = (2+\sqrt{2})\sqrt{(5\sqrt{2}+7)}+(2\sqrt{2}+2)\sqrt{(7+\sqrt{2})}.$$

When we solve this equation regarded as a cubic in $g_{94}+g_{94}^{-1}$ by trial, we obtain Ramanujan's result

$$g_{94}+\frac{1}{g_{94}} = \frac{\sqrt{(5\sqrt{2}+7)}+\sqrt{(7+\sqrt{2})}}{2},$$

A slight modification of this result is

$$g_{94}+g_{94}^{-1} = \tfrac{1}{2}\{\sqrt{2}+1+\sqrt{(6\sqrt{2}-5)}\}\sqrt{(\sqrt{2}+1)}.$$

We shall subsequently need a quadratic equation satisfied by $(g_{94}^3+g_{94}^{-3})/\sqrt{(\sqrt{2}+1)}$. Denoting this expression by $Z$, we immediately find that

$$Z = 3\sqrt{2}+4+(2\sqrt{2}+2)\sqrt{(6\sqrt{2}-5)},$$

and so $$Z^2-(6\sqrt{2}+8)Z-(8\sqrt{2}+2) = 0.$$

(xxv) $\mathbf{n = 98 = 2\times 7^2}$. (G. II, class-number 4.)

One of the forms of the modular equation of order 7 given by Ramanujan is equivalent to the result*

$$\frac{G_m^4}{G_{49m}^4}+\frac{G_{49m}^4}{G_m^4}+7 = 2\sqrt{2}.\left\{G_m^3\,G_{49m}^3+\frac{1}{G_m^3\,G_{49m}^3}\right\}.$$

* Cf. Schläfli, *Journal für Math.* 72 (1870), 360–9.

If $q$ and $q^7$ are the parameters associated with $G_m$ and $G_{49m}$, and we regard this equation as an identity connecting functions of $q$, then the result* of replacing $q$ throughout the identity by $-q$ is the modular equation

$$\frac{g_m^4}{g_{49m}^4}+\frac{g_{49m}^4}{g_m^4}-7 = 2\sqrt{2}.\left\{g_m^3 g_{49m}^3+\frac{1}{g_m^3 g_{49m}^3}\right\}.$$

Now take $m = 2$; since $g_2 = 1$, we have

$$g_{98}^4+g_{98}^{-4}-7 = 2\sqrt{2}.(g_{98}^3+g_{98}^{-3}).$$

Let $g_{98}+g_{98}^{-1} = \theta$; then

$$\theta^4-4\theta^2-5 = 2\sqrt{2}.(\theta^3-3\theta),$$

that is to say,
$$(\theta^2-\theta\sqrt{2}-3)^2 = 14,$$

and so
$$\theta^2-\theta\sqrt{2}-3 = +\sqrt{14},$$

since (because $\theta > 2$) we obviously have

$$\theta^2-\theta\sqrt{2} > 4-2\sqrt{2} > 0 > 3-\sqrt{14}.$$

When we solve the quadratic in $\theta$, we immediately obtain Ramanujan's result

$$g_{98}+\frac{1}{g_{98}} = \frac{\sqrt{2}+\sqrt{(14+4\sqrt{14})}}{2}.$$

(xxvi) $\mathbf{n = 158.}$ (G. II, class-number 4.)

Ramanujan's method of determining $g_{158}$ is obscured by his not having recorded any form of the modular equation of order 79. His knowledge of such an equation is indicated not so much by his determination of $g_{158}$ as by his construction of the quintic equation satisfied by $G_{79}-G_{79}^{-1}$, which is given in his note-book; this quintic is, of course, also derivable from Russell's† quintics for $G_{79}$ and $G_{79}^3$, of which he was apparently ignorant.

In order to determine $g_{158}$ I found it necessary to construct a form of the modular equation of order 79, and I used what seems to have been Ramanujan's standard method of constructing modular equations; for reasons which will appear presently, I think it likely that Ramanujan actually used some other equation, though I have not succeeded in constructing any equation from which $g_{158}$ can be determined in a more direct or more plausible manner.

* Of course Ramanujan is not unique in adopting this device for constructing new forms of modular equations. See Weber, *Algebra*, 3 (1908), 264.

† *Proc. London Math. Soc.* (1), 21 (1891), 359.

If $P$ denotes $$\sqrt{\left(\frac{1+\sqrt{\{\alpha\beta\}}+\sqrt{\{(1-\alpha)(1-\beta)\}}}{2}\right)},$$

the equation which I have constructed is

$$P+\sqrt[8]{\{\alpha\beta\}}+\sqrt[8]{\{(1-\alpha)(1-\beta)\}}+\sqrt[8]{\{\alpha\beta(1-\alpha)(1-\beta)\}}-$$
$$-\tfrac{1}{2}[1+\sqrt[4]{\{\alpha\beta\}}+\sqrt[4]{\{(1-\alpha)(1-\beta)\}}]$$
$$=\sqrt{[P+\sqrt[4]{\{\alpha\beta\}}]}.\sqrt[16]{\{(1-\alpha)(1-\beta)\}}+\sqrt{[P+\sqrt[4]{\{(1-\alpha)(1-\beta)\}}]}.\sqrt[16]{\{\alpha\beta\}}\pm$$
$$\pm\sqrt{[P+1]}.\sqrt[16]{\{\alpha\beta(1-\alpha)(1-\beta)\}}.$$

When the upper sign is taken, this equation is satisfied both when $\beta$ is of degree 47 in $\alpha$, and also when $\beta$ is of degree 79 in $\alpha$; when the lower sign is taken, it is satisfied when $\beta$ is of degree 127 in $\alpha$.

This equation is to be compared with Fiedler's* equations of orders 47 and 79 and Hanna's† equations of orders 47 and 127; I have not attempted to establish any connexion between them.

When we make the standard substitutions

$$\alpha\beta = X^{16}, \qquad (1-\alpha)(1-\beta) = 4X^8,$$

with $$X^4+X^{-4} = (g_{2m}^6+g_{2m}^{-8})\sqrt{2},$$

as explained in the initial general discussion of $g_{2m}$, we find that $g_{94}$, $g_{158}$, and $g_{254}$ are derivable from one or other of the equations

$$\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)\left[X^2+\frac{1}{X^2}+\sqrt{2}+2^{\frac{1}{4}}\left(X+\frac{1}{X}\right)\right]$$
$$=\frac{2^{\frac{1}{4}}}{X^{\frac{1}{2}}}\sqrt{\left(\frac{X^2+X^{-2}}{\sqrt{2}}+X^2\right)}\mp 2^{\frac{1}{4}}X^{\frac{1}{2}}\sqrt{\left(\frac{X^2+X^{-2}}{\sqrt{2}}+\frac{1}{X^2}\right)}.$$

The rationalized form of the equations

$$A = \sqrt{B}\mp\sqrt{C}$$

being $$A^4-2A^2(B+C)+(B-C)^2 = 0,$$

we find on reduction that the rationalized form of the equations associated with $g_{94}$, $g_{158}$, and $g_{254}$ is

$$y^8-(16+8\sqrt{2})2^{\frac{1}{4}}y^7+(120+104\sqrt{2})y^6+(16-32\sqrt{2})2^{\frac{1}{4}}y^5-$$
$$-(1924+1480\sqrt{2})y^4-(96-80\sqrt{2})2^{\frac{1}{4}}y^3+(7312+5392\sqrt{2})y^2+$$
$$+(96-64\sqrt{2})2^{\frac{1}{4}}y-(8892+6384\sqrt{2}) = 0,$$

where $y$ has been written in place of $X+X^{-1}$.

* *Vierteljahrsschrift der naturforschenden Ges. in Zürich*, 30 (1885), 129–229.

† *Proc. London Math. Soc.* (2), 28 (1928), 46–52.

This octavic equation breaks up into the following three equations:

$$y^2-(2+\sqrt{2})2^{\frac{1}{4}}y-(6+3\sqrt{2}) = 0,$$

$$y^2-(4+2\sqrt{2})2^{\frac{1}{4}}y-(12+5\sqrt{2}) = 0,$$

$$y^4-(10+5\sqrt{2})2^{\frac{1}{4}}y^3+(2+10\sqrt{2})y^2+(38+18\sqrt{2})2^{\frac{1}{4}}y-38(1+\sqrt{2}) = 0.$$

The first of the three was to have been anticipated because it is associated with $g_{94}$; when it is removed from the octavic equation, the factorization of the resulting sextic is not very evident without a knowledge of $g_{158}$, and it is this which gives the impression that Ramanujan probably obtained an equation for $g_{158}$ isolated from any equation for $g_{254}$.

The second quadratic has one positive root (10·0231 nearly); the quartic has one positive root (19·2732 nearly) and one negative root (−1·9987 nearly), its other roots being complex.

The equations

$$y = X+X^{-1}, \qquad X^4+X^{-4} = (g_{2m}^6+g_{2m}^{-6})\sqrt{2}, \qquad g_{2m} > 1$$

make $y$ increase as $m$ increases when $m > 1$, since $g_{2m}$ increases with $m$ when $m > 1$; and so the positive root of the second quadratic appertains to $g_{158}$, while the positive root of the quartic appertains to $g_{254}$.

The second quadratic is equivalent to

$$X^2+X^{-2}-(10+5\sqrt{2}) = 2^{\frac{1}{4}}(4+2\sqrt{2})(X+X^{-1})$$

and hence, by squaring,

$$X^4+X^{-4}+88+52\sqrt{2} = (52+34\sqrt{2})(X^2+X^{-2}).$$

The standard transformation now gives

$$g_{158}^6+g_{158}^{-6} = 1716+1210\sqrt{2}\pm(136+104\sqrt{2})\sqrt{(73+52\sqrt{2})},$$

and the upper sign is to be taken. Hence

$$\begin{aligned} g_{158}^3+g_{158}^{-3} &= \sqrt{(894+626\sqrt{2})}+\sqrt{(824+584\sqrt{2})} \\ &= (12+10\sqrt{2})\sqrt{(1+\sqrt{2})}+(6+5\sqrt{2})\sqrt{(9+\sqrt{2})}. \end{aligned}$$

When we regard the last result as a cubic equation to determine $g_{158}+g_{158}^{-1}$ and solve it by trial, we obtain Ramanujan's result

$$g_{158}+\frac{1}{g_{158}} = \frac{\sqrt{(13\sqrt{2}+17)}+\sqrt{(9+\sqrt{2})}}{2}.$$

A slight modification of this result is

$$g_{158}+g_{158}^{-1} = \tfrac{1}{2}\{2\sqrt{2}+1+\sqrt{(8\sqrt{2}-7)}\}\sqrt{(\sqrt{2}+1)}.$$

We shall subsequently need a quadratic equation satisfied by

$(g_{158}^3+g_{158}^{-3})/\sqrt{(\sqrt2+1)}$. Denoting this expression by $Z$, we immediately find that

$$Z = 10\sqrt2+12+(5\sqrt2+6)\sqrt{(8\sqrt2-7)},$$

and so

$$Z^2-(20\sqrt2+24)Z-(28\sqrt2+14) = 0.$$

(xxvii) $\mathbf{n = 254.}$ (G. II, class-number 8.)

I conclude this paper with the investigation of $g_{254}$, a singular modulus which has not been calculated by Ramanujan, nor indeed, so far as I know, by anyone else.

The quartic appertaining to $g_{254}$ which occurred in the course of the discussion of $g_{158}$, namely

$$y^4-(10+5\sqrt2)2^{\frac14}y^3+(2+10\sqrt2)y^2+(38+18\sqrt2)2^{\frac14}y-38(1+\sqrt2) = 0,$$

is not resoluble into two quadratics with coefficients containing no irrationalities other than $2^{\frac14}$. It is, of course, possible to obtain a quartic in $X^4+X^{-4}$, and so in $g_{254}^5+g_{254}^{-6}$, by writing $X+X^{-1}$ for $y$ and then transposing and squaring twice. Such a quartic is not very well adapted for reduction by direct means; but it is possible to obtain a quartic satisfied by $g_{254}^3+g_{254}^{-3}$ in a different manner, and this last quartic can be manipulated so as to reduce the determination of $g_{254}$ to the solution of a set of quadratic equations.

We return to the pair of equations

$$\left(\frac{1}{\sqrt2}-\frac12\right)\left[X^2+\frac{1}{X^2}+\sqrt2+2^{\frac14}\left(X+\frac1X\right)\right]$$
$$= \frac{2^{\frac14}}{X^{\frac14}}\sqrt{\left(\frac{X^2+X^{-2}}{\sqrt2}+X^2\right)}\mp 2^{\frac14}X^{\frac14}\sqrt{\left(\frac{X^2+X^{-2}}{\sqrt2}+\frac{1}{X^2}\right)},$$

associated with $g_{94}$ and $g_{158}$ (with the upper sign) and with $g_{254}$ (with the lower sign), and we observe that, in each case,

$$\left(\frac{X^2+X^{-2}}{\sqrt2}+X^2\right)\left(\frac{X^2+X^{-2}}{\sqrt2}+\frac{1}{X^2}\right) = \left(\frac{1}{\sqrt2}+\frac12\right)\left(X^2+\frac{1}{X^2}\right)^2+1$$
$$= (1+2^{-\frac12})(g_{2m}^6+g_{2m}^{-6})+(2+\sqrt2)$$
$$= (1+2^{-\frac12})(g_{2m}^3+g_{2m}^{-3})^2.$$

Consequently, if, as hitherto, we write $Z\sqrt{(\sqrt2+1)}$ for $g_{2m}^3+g_{2m}^{-3}$, the pair of equations under consideration transform into

$$\tfrac14(3-2\sqrt2)[X^2+X^{-2}+\sqrt2+2^{\frac12}(X+X^{-1})]^2$$
$$= 2^{-\frac14}(X+X^{-1})(X^2+X^{-2}+\sqrt2)\mp 2Z(\sqrt2+1),$$

by squaring both sides of the original equations.

Now transform the last pair of equations in the following manner:

first, transpose all the odd powers of $X+X^{-1}$ to one side and then square, so that the resulting pair of equations are rational in $Z$ and in $X^2+X^{-2}$; next, observe that

$$\begin{aligned} X^2+X^{-2} &= \sqrt{[2+(g_{2m}^6+g_{2m}^{-6})\sqrt{2}]} \\ &= \sqrt{[(2+\sqrt{2})Z^2+2-2\sqrt{2}]}, \end{aligned}$$

substitute this expression for $X^2+X^{-2}$ in the pair of equations now under consideration, and then transpose and square in such a way as to obtain a pair of equations which are rational in $Z$. The result of carrying out these operations is the pair of equations

$$\begin{aligned} Z^8 \pm (64+48\sqrt{2})Z^7 - (6392+4576\sqrt{2})Z^6 \pm (75648+53536\sqrt{2})Z^5 + \\ + (2\,19224+1\,53024\sqrt{2})Z^4 \pm (1\,44128+1\,02464\sqrt{2})Z^3 + \\ + (7456-24192\sqrt{2})Z^2 \pm (13824+8832\sqrt{2})Z + \\ + (1\,11888-66304\sqrt{2}) = 0. \end{aligned}$$

When the upper signs are taken, we have an equation appertaining to $g_{94}$ and $g_{158}$, and so the left-hand side must have two factors

$$\begin{aligned} Z^2-(8+6\sqrt{2})Z-(2+8\sqrt{2}), \\ Z^2-(24+20\sqrt{2})Z-(14+28\sqrt{2}), \end{aligned}$$

in view of the results which have been obtained concerning $g_{94}$ and $g_{158}$. Taking the lower signs is equivalent to changing the sign of $Z$, and so we infer that the octavic equation appertaining to $g_{254}$ must break up into three equations of which two are the quadratics

$$\begin{aligned} Z^2+(8+6\sqrt{2})Z-(2+8\sqrt{2}) = 0, \\ Z^2+(24+20\sqrt{2})Z-(14+28\sqrt{2}) = 0, \end{aligned}$$

and, by dividing the appropriate octavic by the last two quadratics, it is found that the third of the equations is the quartic

$$\begin{aligned} Z^4-(96+74\sqrt{2})Z^3+(112+20\sqrt{2})Z^2 + \\ + (144-36\sqrt{2})Z+(444-296\sqrt{2}) = 0. \end{aligned}$$

The positive roots of the last two quadratics are so small that they would yield complex values of $g_{2m}$; therefore $g_{254}$ is to be derived from the quartic, which, when regarded as an equation in $g_{254}$, is a reciprocal equation of degree 24. The quartic has two positive roots, which are nearly equal to 199·94787 and 1·18688; the former of these is $(g_{254}^3+g_{254}^{-3})/\sqrt{(\sqrt{2}+1)}$; the other two roots are complex, and they are the roots of the quadratic

$$Z^2+cZ+d = 0,$$

where $c = 0{\cdot}48294\,2$, $d = 0{\cdot}10699\,5$, nearly.

When the quartic in $Z$ is expressed in terms of $g_{254}$, it assumes the form

$$(g^{12}+g^{-12})-(96+74\sqrt{2})(g^9+g^{-9})\sqrt{(\sqrt{2}+1)}+(156+132\sqrt{2})(g^6+g^{-6})-$$
$$-(216+114\sqrt{2})(g^3+g^{-3})\sqrt{(\sqrt{2}+1)}+(458+264\sqrt{2})=0,$$

the suffix 254 being omitted for brevity.

It is not difficult to verify that the expression on the left is the product of the three factors comprised in the expression

$$(\omega^4g^4+\omega^{-4}g^{-4})-(1+2\sqrt{2})(\omega^3g^3+\omega^{-3}g^{-3})\sqrt{(\sqrt{2}+1)}-3(\omega^2g^2+\omega^{-2}g^{-2})-$$
$$-(6+3\sqrt{2})(\omega g+\omega^{-1}g^{-1})\sqrt{(\sqrt{2}+1)}-(4+3\sqrt{2}),$$

where $\omega$ assumes the values 1, $\exp(\frac{2}{3}\pi i)$, $\exp(\frac{4}{3}\pi i)$ in turn; and it is, of course, the first of these factors (i.e. the real factor) which vanishes.

By rearranging this factor we find that

$$(g_{254}+g_{254}^{-1})^4-(1+2\sqrt{2})(g_{254}+g_{254}^{-1})^3\sqrt{(\sqrt{2}+1)}-7(g_{254}+g_{254}^{-1})^2+$$
$$+(3\sqrt{2}-3)(g_{254}+g_{254}^{-1})\sqrt{(\sqrt{2}+1)}-(3\sqrt{2}-4)=0.$$

Hence $(g_{254}+g_{254}^{-1})/\sqrt{(\sqrt{2}+1)}$ is the positive root of the quartic

$$x^4-(1+2\sqrt{2})x^3-(7\sqrt{2}-7)x^2+(9-6\sqrt{2})x-(17\sqrt{2}-24)=0.$$

Since

$$x^4-(1+2\sqrt{2})x^3-(7\sqrt{2}-7)x^2+(9-6\sqrt{2})x-(17\sqrt{2}-24)$$
$$\equiv[x^2-\tfrac{1}{2}(1+2\sqrt{2})x-\tfrac{1}{2}(6\sqrt{2}-5)]^2-\tfrac{1}{4}(x+1)^2(1+8\sqrt{2}),$$

we infer that $(g_{254}+g_{254}^{-1})/\sqrt{(\sqrt{2}+1)}$ is the positive root of the quadratic

$$x^2-\tfrac{1}{2}(1+2\sqrt{2})x-\tfrac{1}{2}(6\sqrt{2}-5)-\tfrac{1}{2}(x+1)\sqrt{(1+8\sqrt{2})}=0,$$

the correct sign being attached to the last group of terms.

The determination of $g_{254}$ is now reduced to the solution of a system of quadratic equations, and there is no point in proceeding further with the reduction.

# SYMBOLIC RELATIONS ASSOCIATED WITH FOURIER TRANSFORMS

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## Introduction

THIS note illustrates a symbolic process which may frequently be employed with advantage in the discussion of a large class of definite integrals with a kernel $K(xt)$. The results are such as are usually obtained by the inversion of the order of integrations in a repeated integral. It is here shown that, granted certain rules of interpretation, a definite integral of this type admits a symbolic form of expression which, after manipulation and reinterpretation, furnishes transformations for discussion by the more rigorous methods of analysis. Such merit as the method may possess resides mainly in the suggestion rather than in the justification of such transformations, and no discussion of the validity of processes and results is here attempted.

There is an evident similarity to the Heaviside method but, whereas the fundamental operation in the latter theory is $D\ (\equiv d/dx)$, I adopt in the present discussion $\delta\ (\equiv xd/dx)$. I thereby simplify operations on power series, and emphasize the interesting fact that many integral transformations are formally equivalent to well-known transformations of gamma functions.

The material selected for purposes of illustration is taken mainly from the theory of Fourier transforms in which the transformations of the fundamental symbolic forms lead naturally to results which have recently been placed on a satisfactory basis by Hardy and Titchmarsh.*

## 1. The operators

The symbolic operators employed are expressed in terms of the elementary operator

$$\delta \equiv xd/dx.$$

They are assumed to obey the known laws of combination and use

* *Quart. J. of Math.* (Oxford), 1 (1930), 196–231, referred to as HT. 1; *Proc. London Math. Soc.* (2), 33 (1931), 225–32, referred to as HT. 2.

which apply to operators which are polynomial in $\delta$. Thus, if $U(\delta)$, $V(\delta)$ are two symbolic operators,

$$U(\delta) \sum a_r x^r = \sum a_r U(r) x^r, \tag{1}$$

$$U(\delta)\, x^r y = x^r U(\delta+r) y, \tag{2}$$

and

$$U[Vy] = V[Uy] = [UV]y. \tag{3}$$

The principal symbolic operators employed are the following:

(*a*) $\Gamma(a\delta+b)$ where $a$, $b$ are constants. Such operators furnish convenient and suggestive 'short-hand' expressions for many well-known series. Thus, for Bessel's function,

$$J_\nu(x) = (\tfrac{1}{2}x)^\nu \{\Gamma(\tfrac{1}{2}\delta+\nu+1)\}^{-1} \exp(-\tfrac{1}{4}x^2). \tag{4}$$

(*b*) $t^\delta$, where $t$ is a parameter independent of $x$.

By (1)

$$t^\delta \sum a_r x^r = \sum a_r (xt)^r,$$

and we may thus define the operation in an extended sense by means of

$$t^\delta f(x) = f(xt). \tag{5}$$

(*c*) The operators $\sin(\frac{1}{2}\pi\delta)$, $\cos(\frac{1}{2}\pi\delta)$, which are also employed, may be defined directly from (1) or alternatively as

$$\left.\begin{aligned} \cos \tfrac{1}{2}\pi\delta &= \pi/[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\delta)] = \tfrac{1}{2}\{i^\delta+(-i)^\delta\} \\ \sin \tfrac{1}{2}\pi\delta &= \pi/[\Gamma(\tfrac{1}{2}\delta)\Gamma(1-\tfrac{1}{2}\delta)] = \tfrac{1}{2}i^{-1}\{i^\delta-(-i)^\delta\} \end{aligned}\right\} \tag{6}$$

where

$$i^2 = -1.$$

The definitions are mutually consistent and from (5), (6) we have

$$\cos \tfrac{1}{2}\pi\delta\, f(x) = \tfrac{1}{2}\{f(ix)+f(-ix)\} \tag{7}$$

$$\sin \tfrac{1}{2}\pi\delta\, f(x) = \frac{1}{2i}\{f(ix)-f(-ix)\}.$$

(*d*) The 'integral' operation.

Suppose that, for a certain range of values of $\lambda$,

$$\int_a^b w(\lambda, t)\, dt = U(\lambda) \tag{8}$$

where the range of integration may be finite or infinite but the limits $a$, $b$ are independent of $t$ and $\lambda$.

Suppose further that

$$w(\delta, t) f(x), \qquad U(\delta) f(x)$$

have meaning.

The integral operation is the assumption that

$$\int_a^b [w(\delta, t)f(x)]\,dt = U(\delta)f(x). \tag{9}$$

If $f(x) = \sum c_\lambda x^\lambda$, where every index lies in the range of values for which (8) holds, then it *may* be possible to justify (9); if $f(x)$ does not fulfil this condition, then, notwithstanding, (9) *may* still be true. From our present standpoint we are to regard (9) as a symbolic analogy or parallel of (8), the manipulation and interpretation of which may lead to theorems concerning more familiar entities.

If, for instance,
$$U(\lambda) = U_1(\lambda)U_2(\lambda)$$
where $U_1$, $U_2$ have suitable integral representations, then there will be for investigation a parallel transformation expressing a repeated integral in terms of a single integral or, alternatively, if $U$ can be factorized in various ways there will be parallel relations between repeated integrals demanding consideration. The succeeding paragraph illustrates these points and furnishes results which will be of use at a later stage.

## 2. First illustrations

If $x$ is positive,

$$F(x) \equiv \int_0^\infty e^{-xt}f(t)\,dt = x^{-1}\int_0^\infty e^{-u}f(u/x)\,du$$

$$= x^{-1}\int_0^\infty u^{-\delta}e^{-u}\,du\,f(1/x)$$

or
$$F(x) \equiv \int_0^\infty e^{-xt}f(t)\,dt = x^{-1}\Gamma(1-\delta)\,f(1/x). \tag{10}$$

If $f(x) = \sum_0^\infty a_r x^r$, then

$$x^{-1}\Gamma(1-\delta)f(1/x) = \sum_0^\infty r!a_r x^{-r-1},$$

a series which, when it converges, has been shown by Hardy* to be the value of the integral.

Writing $x$ for $1/x$ in (10),†

$$F(1/x) = x\Gamma(1+\delta)f(x).$$

* *Cambridge Phil. Trans.*, vol. 21, p. 5.

† This transformation of variable is accompanied by transformation of operators $\delta$ for $-\delta$.

If the operation above considered is twice repeated,

$$\int_0^\infty e^{-xu}\,du \int_0^\infty e^{-ut}f(t)\,dt$$

$$= \int_0^\infty e^{-xu}F(u)\,du = x^{-1}\Gamma(1-\delta)F(1/x)$$

$$= x^{-1}\Gamma(1-\delta)x\Gamma(1+\delta)f(x) = \Gamma(-\delta)\Gamma(1+\delta)f(x)$$

$$= \int_0^\infty \frac{t^\delta}{1+t}\,dt\,f(x)$$

$$= \int_0^\infty \frac{f(xt)}{1+t}\,dt = \int_0^\infty \frac{f(u)}{x+u}\,du.$$

Thus the relation

$$\int_0^\infty e^{-xu}\,du \int_0^\infty e^{-ut}f(t)\,dt = \int_0^\infty \frac{f(u)}{x+u}\,du \tag{11}$$

is the integral parallel of

$$\Gamma(-a)\Gamma(1+a) = B(-a, 1+a).$$

Other relations between gamma functions may be similarly interpreted. The duplication formula may be written as

$$\Gamma(1-a) = \pi^{-\frac{1}{2}}2^{-a}\Gamma(1-\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}a).$$

Now
$$2^{-\delta}f(1/x) = f(2/x),$$

$$\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}\delta)\phi(2/x) = 2\int_0^\infty u^{k-1-\delta}e^{-u^2}\,du\,\phi(2/x)$$

$$= 2\int_0^\infty u^{k-1}e^{-u^2}\phi(2u/x)\,du,$$

and, employing (10), the integral parallel is

$$\int_0^\infty e^{-t}f(t/x)\,dt = 4\pi^{-\frac{1}{2}}\int_0^\infty e^{-u^2}\,du\int_0^\infty ve^{-v^2}f(2uv/x)\,dv, \tag{12}$$

which, for suitable forms of the function $f$, may also be obtained by making the substitutions

$$u-v = \xi, \qquad 2uv = t,$$

in
$$\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}(u+v)f(2uv/x)\,dudv.$$

To apply (12) write it in the form

$$\int_0^\infty e^{-t} f(t/x)\,dt = \pi^{-\frac{1}{2}} \int_0^\infty u^{-\frac{1}{2}} e^{-u}\,du \int_0^\infty e^{-v} f(2\sqrt{uv}/x)\,dv$$

and set, in turn, $f(x) = e^{-k^2x^2}$, $\cos(x^2)$, when, after obvious transformations,

$$\int_0^\infty e^{-xz-k^2z^2}\,dz = (\pi k^2)^{-\frac{1}{2}} \int_0^\infty \frac{e^{-(x^2z^2/4k^2)}}{1+z^2}\,dz,$$

and*

$$\int_0^\infty e^{-xz}\cos(z^2)\,dz = \pi^{-\frac{1}{2}} \int_0^\infty \frac{e^{-(x^2z^2/4)}}{1+z^4}\,dz.$$

The formula for $\Gamma(na)$ leads to a generalization of (12) in which an $n$-fold integral is expressed as a single integral.

Returning to (10): if

$$F(x) = \int_0^\infty e^{-xt} f(t)\,dt = x^{-1}\Gamma(1-\delta) f(1/x),$$

then, on the one hand,

$$f(1/x) = \frac{1}{\Gamma(1-\delta)} xF(x) = x \frac{1}{\Gamma(-\delta)} F(x)$$

or

$$f(x) = x^{-1} \frac{1}{\Gamma(\delta)} F(1/x),$$

while on the other hand, by a well-known formula,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} F(s)\,ds.$$

Comparing results, we have the following rule of interpretation:

$$\frac{1}{\Gamma(\delta)} F(1/x) = \frac{x}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} F(s)\,ds$$

and, more generally,

$$\frac{1}{\Gamma(\delta+a)} F(1/x) = x^{-a} \frac{1}{\Gamma(\delta)} x^a F(1/x)$$

$$= \frac{x^{1-a}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} s^{-a} F(s)\,ds. \qquad (13)$$

* Cauchy, cf. *Encyklopädie*, II, 1, ii, 1153.

For example,

$$\frac{1}{\Gamma(\delta+\nu+1)}e^{-\frac{1}{4}x} = \frac{x^{-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left(xs-\frac{1}{4s}\right)s^{-\nu-1}\,ds$$

and, changing $x$ to $x^2$,

$$\frac{1}{\Gamma(\frac{1}{2}\delta+\nu+1)}e^{-\frac{1}{4}x^2} = \frac{x^{-2\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left(x^2s-\frac{1}{4s}\right)s^{-\nu-1}\,ds.$$

If now we take $x$ real and $x^2s = u$, $x^2c = c'$, we have from (4)

$$J_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}u^{-\nu-1}\exp\left(u-\frac{x^2}{4u}\right)du,$$

a well-known formula for the Bessel's function.*

Before entering on a detailed discussion of the kernels $\cos xt$, $\sin xt$ I give here an instance in which the kernel is no longer of exponential type.

The starting-point is Weber's integral

$$\int_0^\infty \frac{J_\nu(t)}{t^{\nu-\mu+1}}\,dt = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)}.$$

Let

$$g(x) = \int_0^\infty J_\nu(xt)f(t)\,dt$$

$$= x^{-1}\int_0^\infty t^{-\delta}J_\nu(t)\,dt\,f(1/x).$$

Thus, symbolically,

$$g(x) = x^{-1}2^{-\delta}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\delta)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}\delta)}f(1/x), \tag{14}$$

whence $$\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+\tfrac{1}{2}\delta)xg(x) = \Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}\delta)f(2/x),$$

of which the integral interpretation is

$$\int_0^\infty e^{-u^2/x^2}u^{\nu+1}g(u)\,du = (x^2/2)^{\nu+1}\int_0^\infty e^{-\frac{1}{4}x^2u^2}u^\nu f(u)\,du, \tag{15}$$

a formula easily verified when

$$\nu = 0, \qquad f(t) = e^{-\nu^2t^2}.\dagger$$

It clearly is closely associated with the theory of 'Hankel' transforms.

* Watson, *Bessel Functions*, p. 177. † Watson, loc. cit., p. 393.

### 3. Fourier reciprocals

Let $x$ be real and positive, let $f$, $g$ be cosine reciprocals and $F$, $G$ sine reciprocals, so that

$$g(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \cos xt\, f(t)\, dt, \qquad G(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \sin xt\, F(t)\, dt. \quad (16)$$

Now
$$\int_0^\infty \cos xt\, f(t)\, dt = x^{-1} \int_0^\infty \cos t\, f(t/x)\, dt$$
$$= x^{-1} \int_0^\infty t^{-\delta} \cos t\, dt\, f(1/x).$$

while, for a restricted range of values of $a$,

$$\int_0^\infty t^{-a} \cos t\, dt = \Gamma(1-a)\sin \tfrac{1}{2}\pi a.$$

Thus the symbolic formula corresponding to the cosine formula of (16) is
$$g(x) = (2/\pi)^{\frac{1}{2}} x^{-1} \Gamma(1-\delta)\sin \tfrac{1}{2}\pi\delta\, f(1/x). \quad (17)$$

The corresponding sine formula is
$$G(x) = (2/\pi)^{\frac{1}{2}} x^{-1} \Gamma(1-\delta)\cos \tfrac{1}{2}\pi\delta\, F(1/x). \quad (18)$$

The discussion is, in the main, confined to (17). The formula may often be employed with advantage to evaluate or transform an integral. For example, to obtain the Cauchy transformation*

$$I \equiv \int_0^\infty \frac{\cos xt}{1+t}\, dt = \int_0^\infty \frac{ue^{-xu}}{1+u^2}\, du,$$

we have
$$I = x^{-1}\Gamma(1-\delta)\sin \tfrac{1}{2}\pi\delta\, \frac{x}{1+x}$$
$$= x^{-1}\Gamma(1-\delta)\frac{1}{2i}\left(\frac{ix}{1+ix} + \frac{ix}{1-ix}\right)$$
$$= x^{-1}\Gamma(1-\delta)\frac{x}{1+x^2} = \int_0^\infty \frac{ue^{-xu}}{1+u^2}\, du,$$

by (10).

I now discuss the properties of (17) in some detail. In the first place, the relation between $f$, $g$ is reciprocal. For

$$(2/\pi)^{\frac{1}{2}} x^{-1}\Gamma(1-\delta)\sin \tfrac{1}{2}\pi\delta\, g(1/x)$$
$$= (2/\pi)^{\frac{1}{2}} x^{-1}\Gamma(1-\delta)\sin \tfrac{1}{2}\pi\delta (2/\pi)^{\frac{1}{2}} x\Gamma(1+\delta)\sin -\tfrac{1}{2}\pi\delta\, f(x)$$

* *Encyklopädie*, II, 1, ii, 1144.

$$= (2/\pi)\Gamma(-\delta)\Gamma(1+\delta)\sin(\tfrac{1}{2}\pi\delta+\tfrac{1}{2}\pi)\sin-\tfrac{1}{2}\pi\delta\, f(x)$$
$$= \pi^{-1}\sin(-\pi\delta)\Gamma(-\delta)\Gamma(1+\delta)f(x) = f(x).$$

Again, operating on (17) with $\pi\,\mathrm{cosec}\,\frac{1}{2}\pi\delta\, x$,

$$\Gamma(\tfrac{1}{2}\delta)\Gamma(1-\tfrac{1}{2}\delta)xg(x) = (2\pi)^{\frac{1}{2}}\Gamma(1-\delta)f(1/x),$$

which may also be written

$$\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\delta)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)g(x) = (2\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)f(1/x), \tag{19}$$

which, interpreted in terms of integrals, is

$$x\int_0^\infty \frac{g(t)\,dt}{x^2+t^2} = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\int_0^\infty e^{-xt}f(t)\,dt. \tag{20}$$

This may be found in the literature.*

An interesting application is to take

$$f(x) = e^{-ax}x^{\mu-1}, \qquad g(x) = (2/\pi)^{\frac{1}{2}}\Gamma(\mu)(x^2+a^2)^{-\mu/2}\cos\{\mu\tan^{-1}(x/a)\}$$

with $a > 0$, $\mu > 1$, $x > 0$.

We then have

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{\mu-2}\theta\cos\mu\theta}{x^2+a^2\tan^2\theta}\,d\theta = \tfrac{1}{2}\pi a^{\mu-1}x^{-1}(x+a)^{-\mu}.$$

Again, employing the duplication formula, we may write (17) as

$$g(x) = 2^{\frac{1}{2}}x^{-1}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\delta)}{\Gamma(\frac{1}{2}\delta)}f(2/x)$$

or,
$$x\Gamma(\tfrac{1}{2}\delta+\tfrac{1}{2})\,g(x) = 2^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)f(2/x), \tag{21}$$

the integral parallel being

$$\int_0^\infty e^{-u^2/x^2}g(u)\,du = \frac{x}{\sqrt{2}}\int_0^\infty e^{-\frac{1}{4}x^2u^2}f(u)\,du. \tag{22}$$

In illustration take

$$f(x) = (\pi/2)^{\frac{1}{2}} \quad (x < 1), \qquad = 0 \quad (x > 1)$$
$$g(x) = x^{-1}\sin x$$

when
$$\int_0^\infty e^{-u^2/x^2}\frac{\sin u}{u}\,du = \tfrac{1}{2}\sqrt{\pi}\int_0^x e^{-\frac{1}{4}u^2}\,du.\dagger \tag{23}$$

Reverting to (21) and writing $x\sqrt{2}$ for $x$,

$$x\Gamma(\tfrac{1}{2}\delta+\tfrac{1}{2})g(x\sqrt{2}) = \Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)f(\sqrt{2}/x),$$

* HT. 2, p. 226. † *Encyklopädie*, II, 1, ii. 1142, due to Laplace.

which may be written

$$\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\delta)x^{\frac{1}{2}}g(x\sqrt{2}) = \Gamma(\tfrac{1}{4}-\tfrac{1}{2}\delta)x^{-\frac{1}{2}}f(\sqrt{2}/x)$$

or

$$\psi(x) = \phi(1/x) \tag{24}$$

where

$$\left.\begin{aligned}\phi(x) &= \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\delta)x^{\frac{1}{2}}f(x\sqrt{2})\\ \psi(x) &= \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\delta)x^{\frac{1}{2}}g(x\sqrt{2}).\end{aligned}\right\} \tag{25}$$

The position occupied by the relation

$$\phi(x) = \phi(1/x)$$

in the theory of self-reciprocal functions would thus appear to have its counterpart in the theory of reciprocal functions in general. The usual method of interpretation when applied to (25) gives

$$\phi(x) = 2^{\frac{1}{2}}x^{-\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}u^2/x^2}f(u)\,du,$$

a result due to Hardy and Titchmarsh* who have also given the reciprocal formula expressing $f$ in terms of $\phi$.

The functions satisfying (24) may be introduced in a somewhat different manner, for, if

$$f(x) = \frac{1}{\Gamma(\frac{1}{2}\delta+\frac{1}{2})}x^{-\frac{1}{2}}\phi(x^2/2), \qquad g(x) = \frac{1}{\Gamma(\frac{1}{2}\delta+\frac{1}{2})}x^{-\frac{1}{2}}\psi(x^2/2), \tag{26}$$

then

$$\begin{aligned}&2^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)f(2/x)\\ &\quad= 2^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\delta)}2^{-\frac{1}{2}}x^{\frac{1}{2}}\phi(2/x^2)\\ &\quad= x^{\frac{1}{2}}\phi(2/x^2) = x^{\frac{1}{2}}\psi(x^2/2) = x\Gamma(\tfrac{1}{2}\delta+\tfrac{1}{2})g(x).\end{aligned}$$

Thus the functions (26) satisfy (21) and so (17).

The integral formula is now

$$\phi(x^2/2) = 2x^{-\frac{1}{2}}\int_0^\infty e^{-u^2/x^2}f(u)\,du = \sqrt{2}x^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}x^2u^2}g(u)\,du. \tag{27}$$

## 4. Self-reciprocal functions

For cosine and sine self-reciprocals the formulae (17), (18) take the forms

$$f(x) = (2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)\sin\tfrac{1}{2}\pi\delta\, f(1/x), \tag{28}$$

$$F(x) = (2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)\cos\tfrac{1}{2}\pi\delta\, F(1/x), \tag{29}$$

and $f$ is associated with a function $\phi$, such that

$$\phi(x) = \phi(1/x),$$

* HT. 1, p. 199.

by the relation $$f(x\sqrt{2}) = x^{-\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{4}+\frac{1}{2}\delta)} \phi(x). \tag{30}$$

Certain self-reciprocals may be obtained by expanding $\phi(x)$ in powers of $x$ and performing the operation indicated term by term.

If, for instance, $$\phi(x) = x^{rn}(1+x^{2r})^{-n},$$

we obtain for $f(x)$ multiples of $e^{-\frac{1}{2}x^2}$, $x^{\frac{1}{2}}J_{-\frac{1}{4}}(\frac{1}{2}x^2)$ by taking respectively

$$r = 1,\ n = \tfrac{1}{2}, \qquad r = 2,\ n = \tfrac{1}{4}.$$

Various formulae relating different classes of self-reciprocal functions are known.* From the present point of view such classes are connected by symbolic operations. Thus, for example,

(31) *If* $$F(x) = x^{-1}\Gamma(1-\delta)f(1/x)$$

*and* $f(x)$ *satisfies* (28), *then* $F(x)$ *satisfies* (29). *Similarly, if*

$$f(x) = x^{-1}\Gamma(1-\delta)F(1/x)$$

*and* $F(x)$ *satisfies* (29), *then* $f(x)$ *satisfies* (28).

For $$\begin{aligned}\cos\tfrac{1}{2}\pi\delta\, F(x) &= \sin(\tfrac{1}{2}\pi\delta+\tfrac{1}{2}\pi)x^{-1}\Gamma(1-\delta)f(1/x)\\ &= x^{-1}\Gamma(1-\delta)\sin\tfrac{1}{2}\pi\delta\, f(1/x)\\ &= (\tfrac{1}{2}\pi)^{\frac{1}{2}}f(x).\end{aligned}$$

Thus $$\begin{aligned}&(2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)\cos\tfrac{1}{2}\pi\delta\, F(1/x)\\ &\quad= (2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)(\tfrac{1}{2}\pi)^{\frac{1}{2}}f(1/x)\\ &\quad= x^{-1}\Gamma(1-\delta)f(1/x) = F(x).\end{aligned}$$

The second part of the theorem is proved in a similar manner.

The integral parallel is, by (10):

(32) *If* $f(x)$ *is a cosine* (*sine*) *self-reciprocal, then* $\int_0^\infty e^{-xt}f(t)\,dt$ *is a sine* (*cosine*) *self-reciprocal.**

Again,

(33) *If* $$g(x) = Q(\delta)f(x), \qquad Q(a) = Q(-1-a),$$

*then, if* $f(x)$ *satisfies* (28), *so does* $g(x)$; *with a similar result for* (29).

For $$\begin{aligned}&(2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)\sin\tfrac{1}{2}\pi\delta\, g(1/x)\\ &\quad= (2/\pi)^{\frac{1}{2}}x^{-1}\Gamma(1-\delta)\sin\tfrac{1}{2}\pi\delta\, Q(-\delta)f(1/x)\\ &\quad= (2/\pi)^{\frac{1}{2}}Q(-1-\delta)x^{-1}\Gamma(1-\delta)\sin\tfrac{1}{2}\pi\delta\, f(1/x)\\ &\quad= Q(-1-\delta)f(x) = Q(\delta)f(x) = g(x).\end{aligned}$$

We may satisfy the condition imposed on $Q$ by taking

$$Q(a) = P(a+\tfrac{1}{2}),$$

where $P$ is an even function.

An example of the resulting transformation is given by Hardy and Titchmarsh,* though the restriction imposed by them on the form of $P$ seems unnecessarily stringent.

I here take $$P(\delta+\tfrac{1}{2}) \equiv \sec\left(\frac{\pi\delta}{r}+\frac{\pi}{2r}\right).$$

Then

$$\begin{aligned} P(\delta+\tfrac{1}{2})\,f(x) &= \frac{1}{\pi}\Gamma\left(\frac{r-1-2\delta}{2r}\right)\Gamma\left(\frac{r+1+2\delta}{2r}\right) f(x) \\ &= \frac{1}{\pi}\int_0^\infty \frac{t^{\frac{\delta}{r}-\frac{r-1}{2r}}}{1+t}\,dt\,f(x) \\ &= \frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{r-1}{2r}} f(xt^{1/r})}{1+t}\,dt \\ &= \frac{r}{\pi}\int_0^\infty \frac{(xt)^{\frac{r-1}{2}} f(t)}{x^r+t^r}\,dt. \end{aligned}$$

Thus

(34) *If $f(x)$ is a cosine (sine) self-reciprocal, so is*

$$\int_0^\infty \frac{(xt)^{\frac{r-1}{2}} f(t)}{x^r+t^r}\,dt.$$

The case $r = 1$ is to be found in the literature.†

The operation $Q(\delta)x^{-1}\Gamma(1-\delta)f(1/x)$ of course transforms a cosine self-reciprocal into a sine self-reciprocal.

Taking $$Q(\delta) = \pi^{\frac{1}{2}}/[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\delta)\Gamma(1+\tfrac{1}{2}\delta)]$$

we find, using (14),

$$\begin{aligned} F(x) &\equiv Q(\delta)x^{-1}\Gamma(1-\delta)f(1/x) \\ &= x^{-1}2^{-\delta}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\delta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\delta)}\,f(1/x) \\ &= \int_0^\infty J_0(xt)f(t)\,dt,\dagger \end{aligned}$$

and such examples might be multiplied.

* HT. 1, p. 209. † HT. 2.

# A PROOF OF HADAMARD'S THEOREM AS TO THE MAXIMUM VALUE OF THE MODULUS OF A DETERMINANT

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THIS proof of Hadamard's theorem shows that it is one of a series of inequalities which can be deduced from the theorem that the roots of the secular equation are all real.

**1.** LEMMA. *Given a set of n positive quantities, let $p_r$ be the average value of their products taken r at a time. Then $p_r^2 > p_{r+1}p_{r-1}$ and*

$$p_1 > p_2^{\frac{1}{2}} > p_3^{\frac{1}{3}} > \ldots > p_n^{\frac{1}{n}}.$$

For the roots of the equation

$$f(x,y) \equiv x^n - np_1x^{n-1}y + \frac{n(n-1)}{1.2}p_2x^{n-2}y^2 - \ldots + (-1)^n p_n y^n = 0$$

are real and positive, and, by continued application of Rolle's theorem, so also are the roots of $\dfrac{\partial^{r+s}f}{\partial^r x\,\partial^s y} = 0$. In particular,

$$p_{r-1}x^2 - 2p_r xy + p_{r+1}y^2 = 0$$

has real roots, and $p_r^2 > p_{r-1}p_{r+1}$.

If, then, $p_{r-1}^r > p_r^{r-1}$, we have also

$$p_r^{r+1} = p_r^{2r}/p_r^{r-1} > p_r^{2r}/p_{r-1}^r > p_{r+1}^r;$$

and since $p_1^2 > p_2$, the lemma is proved.

**2.** We now recall the proof that the roots of the secular equation are all real.

Take first a determinant $A$ whose elements are $a_{rs}$ and multiply it, row by row, by the determinant $A'$ whose elements are $a'_{rs}$, where $a_{rs}$ and $a'_{rs}$ are conjugate complex numbers. We obtain a determinant $B = AA'$, whose elements are $b_{rs}$, in which $b_{rs} = \sum_i a_{ri}a'_{is}$, and $b_{rs}$ and $b_{sr}$ are conjugate complex numbers.

Now form from $B$ two new determinants $\phi(x)$ and $\phi(-x)$, by adding or subtracting $x$ to or from the terms of the principal diagonal. Then multiplying $\phi(x)$ and $\phi(-x)$ together, row by row,

we have a determinant whose elements are $c_{rs} = \sum_i b_{ri} b_{is}$ $(r \neq s)$, whilst in the principal diagonal the elements are $c_{rr} - x^2$.

Then, in the expansion of $\phi(x)\phi(-x)$ in powers of $(-x^2)$, it is found that the coefficients are the sums of determinants which are all positive, since they can be expressed as sums of products of conjugate complex numbers, and so we deduce that the values of $x^2$ which satisfy $\phi(x)\phi(-x) = 0$ are real and positive. But the same proof applied to $\phi(-x) = 0$, whose roots are now known to be all real, will show that they also are all positive, since the principal minors of $B$ are also the sums of products of conjugate complex numbers.

Thus the roots of $\phi(-x)$ are all real and positive, and in applying the lemma we have $p_n = B = AA' = |A|^2$ and $np_{n-1}$ is the sum of the squares of the moduli of the first minors of $A$, $\frac{1}{2}n(n-1)p_{n-2}$ of the second minors of $A$, and so on; whilst $np_1$ is the sum of the squares of the moduli of the elements of $A$.

Thus the square root of $n$ multiplied by the $n$th root of the modulus of a determinant (of order $n$) is less than the square root of the sum of the squares of the moduli of the elements of the determinant.

This is not quite Hadamard's theorem, but is equivalent to it, for, if we prepare our determinant $A$ by dividing each element of any row by the square root of the sum of the squares of the moduli of the elements in that row, the theorem takes Hadamard's form.

For determinants whose modulus attains the maximum value 1, the inequalities of the lemma become equalities, and the corresponding 'secular equation', $\phi(-x) = 0$, takes the form $(x-1)^n = 0$.

# ON THE SPHERICALLY SYMMETRIC FIELD IN RELATIVITY

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## 1. Introduction

It has long been known that according to the relativity theory a purely gravitational field, if spherically symmetric, must also be static.* The question arises as to whether the same is true for a field of gravitation and electromagnetism; in particular, can we have spherically symmetric waves in the absence of matter? We shall show that such waves cannot exist and that indeed the field must be a static one; and furthermore we shall find the most general spherically symmetric field permitted by the field equations in the absence of matter.

In classical electrodynamics no account is taken of the gravitational field; the field equations of general relativity, however, contain the gravitational potentials and the electromagnetic potentials together in the same equations, and there is thus, in general relativity, an interaction between the gravitational and the electromagnetic parts of the field, the theory of which interaction has not yet become a part of our intuitional approach towards either electromagnetism or gravitation. In solving the present problem, then, it is evidently not safe to appeal to ordinary intuitions regarding the behaviour of electromagnetism since the very possibility of waves of the type we seek implies, in view of Birkhoff and Eiesland's result, just such an interaction between the electromagnetic and gravitational parts of the field. It is necessary, therefore, to proceed mathematically from the field equations.

## 2. The field equations

The field equations of the general theory of relativity in the absence of matter may be written, by a suitable choice of units, as

$$G_{ab}+E_{ab}=0 \qquad \Phi(ab)$$

and

$$\frac{\partial(F^{ab}\sqrt{-g})}{\partial x^b}=0, \qquad \Phi(a)$$

* G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, p. 253, § 7; and also J. Eiesland, *Trans. American Math. Soc.* 27 (1925), 213.

where $$G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R,$$

$R_{ab}$ is the Ricci tensor of the line element $ds^2 = g_{ab} dx^a dx^b$,

$$E_{ab} = -g^{cd} F_{ac} F_{bd} + \tfrac{1}{4} g_{ab} g^{ce} g^{df} F_{cd} F_{ef}, \tag{1}$$

$$F_{ab} = \frac{\partial \phi_a}{\partial x^b} - \frac{\partial \phi_b}{\partial x^a}, \tag{2}$$

$$g = |g_{ab}|.$$

These field equations are actually the same as those of the projective relativity theory* for the case of zero index and of the recent unified field theory of Einstein and Mayer;† we may therefore regard the present work as the extension of the researches of Birkhoff and Eiesland to the cases of the above unified field theories. Our results also apply to the five-dimensional theory developed by Kaluza and Klein.‡

### 3. Spherical symmetry

The first question to be decided is what is to be meant by a spherically symmetric field in the present case. Eiesland§ has shown that the most general spherically symmetric line element may be reduced to the form

$$ds^2 = A dt^2 - B dr^2 - C(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{3}$$

where $A$, $B$, $C$ are functions of $r$ and $t$ alone; a space having such a line element is moved into itself by the rotations

$$\left.\begin{aligned} U_1 &= \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \\ U_2 &= \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi} \\ U_3 &= \frac{\partial}{\partial\phi} \end{aligned}\right\}, \tag{4}$$

and by spherical symmetry we mean invariance of functional form under these rotations. It is to be noted that neither $r$ nor $t$ appears in (4), so that we may make any transformation of $r$ and $t$ to $R$ and $T$, say, leaving $\theta$ and $\phi$ unchanged, without thereby upsetting the spherical symmetry of any entity according to this definition; we shall make use of this fact later.

* O. Veblen and B. Hoffmann, *Phys. Rev.* 36 (1930), 810.

† A. Einstein and W. Mayer, *Sitz. d. Preuss. Akad.* (Berlin), xxv (1931), p. 541.

‡ Th. Kaluza, *Sitz. d. Preuss. Akad.* (Berlin) (1921), p. 966; O. Klein, *Zeits. f. Phys.* 37 (1926), 895, and 46 (1927), 188.

§ loc. cit., p. 213.

The conditions that the vector potential $\phi_a$ be spherically symmetric are easily found, from the Killing equations

$$\xi^b_{(\sigma)} \frac{\partial \phi_a}{\partial x^b} + \phi_b \frac{\partial \xi^b_{(\sigma)}}{\partial x^a} = 0,$$

where $a$ and $b$ take on the values 1, 2, 3, 4 referring to $r$, $\theta$, $\phi$, $t$ respectively, and

$$\left.\begin{array}{llll} b = & 1, & 2, & 3, & 4 \\ \hline \xi^b_{(1)} = & 0, & \sin\phi, & \cot\theta\cos\phi, & 0 \\ \xi^b_{(2)} = & 0, & \cos\phi, & -\cot\theta\sin\phi, & 0 \\ \xi^b_{(3)} = & 0, & 0, & 1, & 0 \end{array}\right\}, \tag{5}$$

to be that

$$\left.\begin{array}{l} \phi_1 = \phi_1(r,t) \\ \phi_2 = \phi_3 = 0 \\ \phi_4 = \phi_4(r,t) \end{array}\right\}. \tag{6}$$

If the $g_{ab}$ are spherically symmetric, the requirement that $\phi_a$ be spherically symmetric implies that $F_{ab}$ and $E_{ab}$ be also; however, $\phi_a$ is undetermined to the extent of an arbitrary additive gradient, so that the assumption that it be spherically symmetric is certainly not the most general. We might consider the assumption that the field forces $F_{ab}$ be spherically symmetric, the conditions for this being that

$$\xi^c_{(\sigma)} \frac{\partial F_{ab}}{\partial x^c} + F_{ac} \frac{\partial \xi^c_{(\sigma)}}{\partial x^b} + F_{cb} \frac{\partial \xi^c_{(\sigma)}}{\partial x^a} = 0,$$

which, since $F_{ab} = -F_{ba}$, leads to

$$\left.\begin{array}{l} F_{12}, F_{13}, F_{24}, F_{34} = 0 \\ F_{14} = F_{14}(r,t) \\ F_{23} = f(r,t)\sin\theta \end{array}\right\}; \tag{7}$$

but this also is not the most general assumption we could make since, for example, although the spherical symmetry of $F_{ab}$ necessarily implies the spherical symmetry of $E_{ab}$, if $g_{ab}$ be spherically symmetric, nevertheless the spherical symmetry of $E_{ab}$ does not imply the spherical symmetry of $F_{ab}$. *We have, however, a method whereby we may consider the most general case*; for, since the metrical properties of space-time are determined by the $g_{ab}$, we *must* have the $g_{ab}$ spherically symmetric; if we now take a $\phi_a$ upon which no *a priori* conditions of symmetry are imposed, we shall have the most general case. This procedure will enable us to avoid the question of what

is to be meant by the spherical symmetry of the electromagnetic part of the field; we shall see immediately that this question is now, so to speak, decided for us by the field equations $\Phi(ab)$, so far as the present investigation is concerned.

Since $G_{ab}$ is formed invariantively from $g_{ab}$, we shall now have $G_{ab}$ spherically symmetric; the conditions for this are that

$$\xi^c_{(\sigma)} \frac{\partial G_{ab}}{\partial x^c} + G_{ac} \frac{\partial \xi^c_{(\sigma)}}{\partial x^b} + G_{cb} \frac{\partial \xi^c_{(\sigma)}}{\partial x^a} = 0.$$

These conditions are the same as those for the spherical symmetry of $F_{ab}$, but they lead to different results since $G_{ab} = G_{ba}$, whereas $F_{ab} = -F_{ba}$; they give in fact

$$\left.\begin{aligned} G_{12}, G_{13}, G_{23}, G_{24}, G_{34} &= 0, \\ G_{33} &= G_{22} \sin^2\theta, \\ G_{11}, G_{22}, G_{44}, G_{14} &\text{ independent of } \theta \text{ and } \phi. \end{aligned}\right\} \tag{8}$$

We note that we do not find that $G_{14}$ must vanish; the vanishing of $g_{14}$ in (3) is due to the fact that a special choice of the $r$- and $t$-coordinates has been made; actual computation of $G_{ab}$ for the coordinate system in which the line element (3) is expressed shows that $G_{14}$ does not vanish identically in this coordinate system.

The field equations $\Phi(ab)$ now impose upon our *general* electromagnetic field the condition that $E_{ab}$ be spherically symmetric; we shall thus have conditions like (8) satisfied by $E_{ab}$.

## 4. Proof that the gravitational part of the field is static

For the line element and coordinate system of (3) we have

$$\left.\begin{aligned} E_{11} &= \frac{1}{2}\Big\{\frac{1}{C}(F_{12})^2 + \frac{1}{C\sin^2\theta}(F_{13})^2 - \frac{1}{A}(F_{14})^2 - \frac{B}{C^2\sin^2\theta}(F_{23})^2 + \\ &\qquad + \frac{B}{AC}(F_{24})^2 + \frac{B}{AC\sin^2\theta}(F_{34})^2\Big\}, \\ E_{22} &= \frac{1}{2}\Big\{\frac{1}{B}(F_{12})^2 + \frac{1}{C\sin^2\theta}(F_{23})^2 - \frac{1}{A}(F_{24})^2 - \frac{1}{B\sin^2\theta}(F_{13})^2 + \\ &\qquad + \frac{C}{AB}(F_{14})^2 + \frac{1}{A\sin^2\theta}(F_{34})^2\Big\}, \\ E_{33} &= \frac{1}{2}\Big\{\frac{1}{B}(F_{13})^2 + \frac{1}{C}(F_{23})^2 - \frac{1}{A}(F_{34})^2 - \frac{\sin^2\theta}{B}(F_{12})^2 + \\ &\qquad + \frac{C\sin^2\theta}{AB}(F_{14})^2 + \frac{\sin^2\theta}{A}(F_{24})^2\Big\}, \end{aligned}\right\} \tag{9}$$

$$
\left.\begin{aligned}
E_{44} &= \frac{1}{2}\left\{\frac{1}{B}(F_{14})^2+\frac{1}{C}(F_{24})^2+\frac{1}{C\sin^2\theta}(F_{34})^2+\frac{A}{BC}(F_{12})^2+\right.\\
&\qquad\left.+\frac{A}{BC\sin^2\theta}(F_{13})^2+\frac{A}{C^2\sin^2\theta}(F_{23})^2\right\},\\
E_{12} &= \frac{1}{C\sin^2\theta}F_{13}F_{23}-\frac{1}{A}F_{14}F_{24},\\
E_{13} &= -\frac{1}{C}F_{12}F_{23}-\frac{1}{A}F_{14}F_{34},\\
E_{14} &= -\frac{1}{C}F_{12}F_{24}-\frac{1}{C\sin^2\theta}F_{13}F_{34},\\
E_{23} &= \frac{1}{B}F_{12}F_{13}-\frac{1}{A}F_{24}F_{34},\\
E_{24} &= \frac{1}{B}F_{12}F_{14}-\frac{1}{C\sin^2\theta}F_{23}F_{34},\\
E_{34} &= \frac{1}{B}F_{13}F_{14}+\frac{1}{C}F_{23}F_{24}.
\end{aligned}\right\} \quad (9)
$$

From $\Phi(12)$ and $\Phi(34)$ we have, using (8) and (9),

$$\frac{1}{C}F_{14}F_{23}\left\{\frac{1}{B\sin^2\theta}(F_{13})^2+\frac{1}{A}(F_{24})^2\right\}=0;$$

but the functions $A$, $B$, $C$ must always be positive, since the signature of the quadratic form (3) is determined by their signs, and furthermore the quantities $F_{ab}$ must all be real if they are to have physical meaning; hence we must have

$$
\left.\begin{aligned}
&\text{either} && F_{14}=0,\\
&\text{or} && F_{23}=0,\\
&\text{or} && F_{13},F_{24}=0.
\end{aligned}\right\}
$$

Substituting $F_{14}=0$, for example, in $\Phi(12)$ and $\Phi(34)$ we find that

$$
\left.\begin{aligned}
&\text{either} && F_{23}=0,\\
&\text{or} && F_{13},F_{24}=0,
\end{aligned}\right\}
$$

and an analogous result holds when we substitute $F_{23}=0$ in the same equations; we may therefore sum up our various conditions as

$$
\left.\begin{aligned}
&\text{either} && F_{14},F_{23}=0,\\
&\text{or} && F_{13},F_{24}=0,
\end{aligned}\right\} \quad (10)
$$

these following from $\Phi(12)$ and $\Phi(34)$ as consequences of the facts that $A$, $B$, $C$ are positive and $F_{ab}$ real. If we apply the same

considerations to $\Phi(13)$ and $\Phi(24)$ we obtain the further conditions that

$$\left.\begin{array}{ll} \text{either} & F_{12}, F_{34} = 0, \\ \text{or} & F_{14}, F_{23} = 0. \end{array}\right\} \tag{11}$$

Conditions (10) and (11) must be satisfied simultaneously and there are thus only the two possibilities:

$$\left.\begin{array}{ll} \text{I.} & F_{12}, F_{34}; F_{13}, F_{24} = 0. \\ \text{II.} & F_{14}, F_{23} = 0. \end{array}\right\}$$

We consider these two cases in turn:

CASE I. Here $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$ are the only non-zero components of $F_{ab}$; we thus have, from (9),

$$E_{14} = 0,$$

$$E_{11} = \frac{1}{2}\left\{-\frac{1}{A}(F_{14})^2 - \frac{B}{C^2\sin^2\theta}(F_{23})^2\right\},$$

and

$$E_{44} = \frac{1}{2}\left\{\frac{1}{B}(F_{14})^2 + \frac{A}{C^2\sin^2\theta}(F_{23})^2\right\},$$

so that, from $\Phi(14)$, $\Phi(11)$, $\Phi(44)$, we obtain

$$G_{14} = 0 \text{ and } G_1^1 = G_4^4.$$

But these are precisely the special conditions obtained by Eiesland* under which (3) is reducible to the static form by a suitable transformation of the $r$- and $t$-coordinates, and therefore Case I leads to a static line element.

CASE II. Here $F_{14}$ and $F_{23}$ vanish; from $\Phi(22)$ and $\Phi(33)$, using (8) and (9), we have

$$\frac{1}{B}(F_{12})^2 + \frac{1}{A\sin^2\theta}(F_{34})^2 = \frac{1}{A}(F_{24})^2 + \frac{1}{B\sin^2\theta}(F_{13})^2, \tag{12}$$

which we write as

$$\left.\begin{array}{l} \dfrac{1}{B}(F_{12})^2 + \dfrac{1}{A\sin^2\theta}(F_{34})^2 = \Omega \\ \dfrac{1}{A}(F_{24})^2 + \dfrac{1}{B\sin^2\theta}(F_{13})^2 = \Omega \end{array}\right\}. \tag{12a}$$

Now $\Phi(23)$ gives

$$A^2(F_{12})^2(F_{13})^2 = B^2(F_{24})^2(F_{34})^2,$$

* loc. cit., p. 227, Theorem V, Corollary.

and substituting for $(F_{12})^2$ and $(F_{24})^2$ from (12a) into this equation, we have

$$BA^2(F_{13})^2\left\{\Omega-\frac{1}{A\sin^2\theta}(F_{34})^2\right\}=AB^2(F_{34})^2\left\{\Omega-\frac{1}{B\sin^2\theta}(F_{13})^2\right\};$$

the quartic terms in $F_{ab}$ cancel each other and we are left with

$$\Omega\{A(F_{13})^2-B(F_{34})^2\}=0;$$

thus, either $\Omega=0$

or $A(F_{13})^2=B(F_{34})^2$ $\Big\}$.

If we take $\Omega=0$ and wish $F_{ab}$ to be real, (12a) shows that we must have $F_{12}$, $F_{34}$, $F_{13}$, $F_{24}$ all zero, and since we already have $F_{14}$ and $F_{23}$ zero we are reduced to the case of a purely gravitational spherically symmetric field which we know must be static.

We now consider the remaining possibility, namely, that

$$A(F_{13})^2=B(F_{34})^2, \tag{13}$$

which, with (12), implies that

$$A(F_{12})^2=B(F_{24})^2 \tag{14}$$

also. By means of (13) and (14) we may eliminate $F_{24}$ and $F_{34}$ from the expressions (9) for the components of $E_{ab}$; in taking the square root of (13) and (14) we must take the same sign in either case since otherwise $\Phi(23)$ will not be satisfied; we thus have

$$\left.\begin{aligned}E_{11}&=\frac{1}{C}\left\{(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2\right\},\\ E_{44}&=\frac{A}{BC}\left\{(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2\right\},\\ E_{14}&=\pm\frac{1}{C}\sqrt{\frac{A}{B}}\left\{(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2\right\},\end{aligned}\right\}$$

the other components being zero. Thus from the conditions of spherical symmetry of $E_{ab}$ we have merely that

$$K\equiv(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2$$

is independent of $\theta$ and $\phi$; this, together with (13) and (14), constitutes all the information that can be obtained in the present case from the spherical symmetry of $E_{ab}$. For further progress we must make use of the field equations $\Phi(a)$ and the equations (2); the latter introduce the quantities $\phi_a$ which are not uniquely determinate, and

it is more convenient to replace these equations by the equivalent conditions upon $F_{ab}$, namely,

$$\frac{\partial F_{bc}}{\partial x^a}+\frac{\partial F_{ca}}{\partial x^b}+\frac{\partial F_{ab}}{\partial x^c}=0. \qquad \Phi(abc)$$

When we consider the sets of equations $\Phi(a)$ and $\Phi(abc)$ for the present case and express them, by means of (13) and (14), in terms of $F_{12}$ and $F_{13}$ only, we find that $\Phi(1)$ and $\Phi(4)$ each become

$$\sin\theta\frac{\partial}{\partial\theta}\{\sin\theta F_{12}\}+\frac{\partial F_{13}}{\partial\phi}=0 \qquad \Phi(1)$$

and that $\Phi(123)$ and $\Phi(234)$ each become

$$\frac{\partial F_{13}}{\partial\theta}-\frac{\partial F_{12}}{\partial\phi}=0. \qquad \Phi(123)$$

Since $K\equiv(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2$ is independent of $\theta$ and $\phi$, we get, using

$$F_{12}=\sqrt{K-\frac{1}{\sin^2\theta}(F_{13})^2}$$

to eliminate $F_{12}$ from $\Phi(1)$ and $\Phi(123)$,

$$\left.\begin{aligned} F_{13}\frac{\partial F_{13}}{\partial\theta}-\sqrt{K-\frac{1}{\sin^2\theta}(F_{13})^2}\frac{\partial F_{13}}{\partial\phi}-K\sin\theta\cos\theta=0 \\ \text{and}\qquad \sin^2\theta\sqrt{K-\frac{1}{\sin^2\theta}(F_{13})^2}\frac{\partial F_{13}}{\partial\theta}+F_{13}\frac{\partial F_{13}}{\partial\phi}=0. \end{aligned}\right\}$$

If $K$ be zero, we must have, from the definition of $K$, that the $F_{ab}$ be all zero, and we shall thus arrive back at the purely gravitational case once more; so, taking $K$ to be different from zero, we may solve the above equations for $\frac{\partial F_{13}}{\partial\theta}$ to give

$$\frac{\partial F_{13}}{\partial\theta}=F_{13}\cot\theta,$$

which means that

$$F_{13}=L\sin\theta, \qquad (15)$$

$L$ being independent of $\theta$. Since $\left\{(F_{12})^2+\frac{1}{\sin^2\theta}(F_{13})^2\right\}$ and $L$ are each independent of $\theta$, we must thus have $F_{12}$ independent of $\theta$; from $\Phi(1)$ we now obtain

$$\frac{\partial L}{\partial\phi}+F_{12}\cos\theta=0,$$

which therefore means that $F_{12}$ must be zero; and then from (15), (13), and (14) it follows that all components of $F_{ab}$ are zero and we are back to the purely gravitational case.

We have thus shown that a spherically symmetric field of gravitation and electromagnetism in the relativity theory must be such that the gravitational part is static.

## 5. The most general spherically symmetric field outside matter

It is now a simple matter to obtain the most general spherically symmetric field of gravitation and electricity in the absence of matter. For, since the line element must be static, we can find a transformation of coordinates of the type

$$\left.\begin{aligned} r &\to r(R, T) \\ t &\to t(R, T) \\ \theta &\to \theta \\ \phi &\to \phi \end{aligned}\right\} \tag{16}$$

such that it takes the form

$$ds^2 = \mathfrak{A}dT^2 - \mathfrak{B}dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are functions of $R$ alone.* There will be no confusion if we use $r$ and $t$ instead of $R$ and $T$, and $A$ and $B$ instead of $\mathfrak{A}$ and $\mathfrak{B}$, if we remember that the quantities $r$, $t$, $A$, and $B$ are no longer the same as the quantities $r$, $t$, $A$, and $B$ that we used previously; we thus take the line element to be, in the new coordinate system, of the form

$$ds^2 = Adt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{17}$$

where $A$ and $B$ are functions of $r$ alone. As we pointed out at an earlier stage, the transformation (16) did not affect the conditions for the spherical symmetry of the line element, and therefore the components of $G_{ab}$ in the new coordinate system must still satisfy the conditions (8); they must now evidently satisfy the further condition that they are all to be independent of the new coordinate $t$ and it also turns out that the new $G_{14}$ is identically zero, but we shall not have occasion to make specific use of these additional facts. In

* The transformation in question being merely one step beyond that considered by Eiesland, loc. cit., p. 227.

view of the invariance of the field equations $\Phi(ab)$ the components of $E_{ab}$ in the new coordinate system must also still satisfy conditions like (8), and so, proceeding as before from the equations $\Phi(ab)$, $\Phi(a)$, $\Phi(abc)$, (8), and (9) (and substituting $r^2$ wherever $C$ occurs), we shall find that in the new coordinate system, as in the old, the only possible non-zero components of $F_{ab}$ are $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$. Thus the non-vanishing components of $E_{ab}$ in this coordinate system are given by

$$\left.\begin{aligned} E_{11} &= -\tfrac{1}{2}B\Lambda, \\ E_{33} &= E_{22}\sin^2\theta = \tfrac{1}{2}r^2\Lambda\sin^2\theta, \\ E_{44} &= \tfrac{1}{2}A\Lambda, \end{aligned}\right\} \tag{18}$$

where

$$\left.\Lambda = \frac{1}{AB}(F_{14})^2 + \frac{1}{r^4\sin^2\theta}(F_{23})^2.\right\}$$

The equations $\Phi(a)$ and $\Phi(abc)$ in the present case are immediately integrable and lead directly to the results that

$$\left.\begin{aligned} F_{14} &= \frac{\epsilon}{r^2}\sqrt{AB} \\ \text{and} \qquad F_{23} &= \mu\sin\theta \end{aligned}\right\},$$

where $\epsilon$ and $\mu$ are constants of integration, and with these expressions for $F_{14}$ and $F_{23}$ we have

$$\Lambda = \frac{\epsilon^2+\mu^2}{r^4}. \tag{19}$$

Now the field caused by a charged sphere is obtained by postulating that the field be static and spherically symmetric, and that $F_{14} = -F_{41}$ be the only non-zero components of $F_{ab}$ in the coordinate system of equation (17); the work then proceeds exactly as would the above except that instead of $\Lambda$ being given by (19) it is given by

$$\Lambda = \frac{\epsilon'^2}{r^4};$$

hence we shall obtain in the present case precisely the same gravitational part of the field as is obtained for the case of a charged sphere except that instead of $\epsilon'^2$ we must write $\epsilon^2+\mu^2$. Quoting the well-known result, then, for the gravitational part of the field of a charged sphere,* we have as the most general spherically symmetric field of

* See A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd edition, p. 185. The field of a charged sphere was first given by H. Weyl, *Ann. der Physik*, 54 (1917), 117.

gravitation and electromagnetism in the absence of matter

$$\left.\begin{aligned} ds^2 = \left(1-\frac{2m}{r}+\frac{\epsilon^2+\mu^2}{2r^2}\right)dt^2 - \frac{1}{\left(1-\frac{2m}{r}+\frac{\epsilon^2+\mu^2}{2r^2}\right)}\,dr^2 - \\ -r^2(d\theta^2+\sin^2\theta\,d\phi^2), \end{aligned}\right\} \quad (20)$$

with $$F_{14} = \frac{\epsilon}{r^2}$$

and $$F_{23} = \mu\sin\theta.$$

## 6. Interpretation of the field

Let us consider the significance of $\mu$ in (20); so long as we leave the time axis unaltered we may break up the electromagnetic part of the field into electric and magnetic parts, the former being, in the present case, described by $F_{14}$ and the latter by $F_{23}$; $\epsilon$ is thus a parameter describing the strength of the electric part of the field, and $\mu$ one describing the strength of the magnetic part; the components of $F_{ab}$ have no invariant significance in themselves, and to interpret them physically we must form invariants out of them; such an invariant is the absolute magnitude, $\sqrt{F^{ab}F_{ab}}$, of the tensor $F_{ab}$, and in the present case this becomes

$$\sqrt{2\left(-\frac{\epsilon^2}{r^4}+\frac{\mu^2}{r^4}\right)}.$$

For $\mu = 0$ this gives a space-like magnitude $\frac{\epsilon\sqrt{2}}{r^2}$ for the force of the electric field, and for $\epsilon = 0$ we obtain a time-like magnitude $\frac{\mu\sqrt{2}}{r^2}$ for the force of the magnetic field. For the values of $m$ and $\epsilon$ that occur in nature, and for values of $\mu$ of the same order, the quantity $r$ may be taken to be the same as the geodesic distance to a very high degree of accuracy; we may therefore interpret $\mu$ as a measure of the magnetic pole strength of the matter at the origin that would produce the external field (20), and the field itself as that which could be caused by a spherically symmetric (not necessarily static) distribution of matter around the origin such that the total mass is $m$, the total charge $\epsilon$, and the total magnetic pole strength $\mu$. Isolated magnetic poles have not been observed in nature, so that the existence of a distribution of matter that would produce a resulting magnetic pole strength would appear to be improbable; there is some

quantum-mechanical evidence for the possibility of the existence of isolated magnetic poles,* but these poles are of a somewhat different nature from those discussed here. It need not surprise us that we obtain a field that does not occur in nature; in the ordinary Schwarzschild line element the mass of the field-producing body arises mathematically as a constant of integration and may therefore take on negative values and yet leave the field equations satisfied.

## 7. Conclusion

By working with the most general electromagnetic part of the field we have shown that no spherically symmetric gravitational and electromagnetic waves can exist other than the 'coordinate waves' that may always be introduced by a suitable transformation of coordinates and have no objective significance; we have shown, moreover, that it is impossible for a spherically symmetric field outside matter to exist unless it be a static field.

The Schwarzschild line element corresponds to the external gravitational effect of a spherically symmetric distribution of matter having constant total mass; Birkhoff's theorem has as corollary that a corresponding line element does not exist for the case in which the total mass is not constant; we have as an immediate consequence of the present work the result that if the matter have in addition to total mass a total electric charge, then both the total mass and the total charge must be constant; it is thus not possible, according to general relativity, to have an isolated spherically symmetric distribution of matter whose total mass or total charge fluctuates.

* P. A. M. Dirac, *Proc. Roy. Soc.* (London), A. 821, p. 60.

# SINGULAR SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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THE singular solution of the differential equation

$$\phi(x, y, p) = 0, \tag{1}$$

as obtained either from the $c$-discriminant of the general solution $f(x, y, c) = 0$ or from the $p$-discriminant of $\phi(x, y, p) = 0$ itself, may, we know, be accompanied by extraneous factors corresponding respectively to a nodal locus or a tac-locus. Even a factor which appears in both discriminants is not necessarily a singular solution, since a cuspidal locus reckons both as a nodal locus and as a tac-locus. Further tests are therefore necessary.*

If, however, we consider the general solution in the parametric form

$$x = x(p, c), \qquad y = y(p, c), \tag{2}$$

which occurs naturally in certain standard methods of solution, we can determine the singular solution unencumbered with irrelevant factors.†

The singular solution, being the envelope of the family of curves comprised by the general solution, will touch each curve of the family at a point (or points), and the aggregate of these points sufficiently constitutes the envelope. Suppose, then, that the envelope touches a curve $c =$ constant at the point given parametrically by

$$p = p(c). \tag{3}$$

Then the envelope is itself defined parametrically by

$$x = x\{p(c), c\}, \qquad y = y\{p(c), c\}. \tag{4}$$

On a curve $c =$ constant we have, by definition of $p$,

$$dy = p\,dx,$$

i.e.

$$\frac{\partial y}{\partial p} = p\,\frac{\partial x}{\partial p}. \tag{5}$$

* Compare, for instance, Forsyth, *Differential Equations* (1929), 35–47; Piaggio, *Differential Equations* (1921), 65–78; Ince, *Ordinary Differential Equations* (1927), 82–91.

† Strictly speaking, there may be constant factors which are readily recognized as irrelevant.

On the envelope we have

$$dx = \frac{\partial x}{\partial p}\,dp + \frac{\partial x}{\partial c}\,dc, \qquad dy = \frac{\partial y}{\partial p}\,dp + \frac{\partial y}{\partial c}\,dc, \tag{6}$$

where the ratio $dp/dc$ is determined from (3). Since, by hypothesis, the envelope touches each curve, the $dx$, $dy$ of (6) must also satisfy $dy = p\,dx$, and we have, in virtue of (5),

$$\frac{\partial y}{\partial c} = p\,\frac{\partial x}{\partial c}. \tag{7}$$

This is a relation between $p, c$ which, associated with the parametric equations (2), gives a single curve (or at least a finite set of curves). If we retrace the steps of our argument, we see that this curve, wherever it meets a curve $c =$ constant, has the same slope as that curve. This curve (7) is therefore an envelope of (some or all) the curves $c =$ constant or is itself one of those curves. The second alternative corresponds to factors of (7) independent of $p$ which can be recognized and rejected. After such rejection, (7) gives the singular solution free of encumbrances.

We have avoided cusps, since the direction of the curves $c =$ constant has been defined, not by the ratio* $x_p/y_p$, which becomes indeterminate at a cusp, but by $p$ itself, which presumably should be effective at every point. Direction on (7), however, is still defined by analogous ratio: at cusps on (7), therefore, it is possible that contact may fail. But these points are finite in number and do not vitiate the envelope-property of (7).

Elimination of $p$ between (5) and (7) gives

$$\frac{\partial(x, y)}{\partial(p, c)} = 0, \tag{8}$$

the discriminant of the parametric equations (4). This again defines the singular solution but fails to exclude a cuspidal locus, since $x_p = 0 = y_p$, the condition for a cusp on $c =$ constant, satisfies (5) and therefore (8).

We can see also how the nodal locus enters the $c$-discriminant of $f(x, y, c) = 0$, if we substitute the parametric forms (4) into $f = 0$. By hypothesis the equation is satisfied for every $p, c$ and we may

* I have written $x_p$ for $\partial x/\partial p$ and so forth.

differentiate in each of them partially, getting

$$f_x\frac{\partial x}{\partial p}+f_y\frac{\partial y}{\partial p}=0,$$

$$f_x\frac{\partial x}{\partial c}+f_y\frac{\partial y}{\partial c}+f_c=0.$$

Thus $f_c=0$ gives the nodal condition

$$f_x=0=f_y \tag{9}$$

as an alternative to (8).

A similar substitution in the differential equation $\phi(x,y,p)=0$ shows that the $p$-discriminant gives

$$\phi_x=0=\phi_y \tag{10}$$

as an alternative to (8). This is the condition for nodes on $\phi(x,y,\text{const.})=0$, and therefore for tac-points on $f(x,y,c)=0$.

As a simple example, consider the parametric equations

$$x=3p^2+c^2, \qquad y=2p^3+c^3.$$

For the singular solution we have from (7), after removing a constant factor,

$$2p=3c,$$

which gives the semi-cubical parabola

$$x=\frac{31}{4}c^2, \qquad y=\frac{31}{4}c^3$$

for the envelope. The $(p,c)$-discriminant (8) gives

$$p(2p-3c)=0,$$

where the additional factor $p$ represents the cuspidal locus

$$x=c^2, \qquad y=c^3.$$

The equations $f=0$, $\phi=0$ are respectively

$$4(x-c^2)^3=27(y-c^3)^2, \qquad (x-3p^2)^3=(y-2p^3)^2.$$

Their discriminants give the same three curves, namely

$(f=0)$ (i) $c=0$; (ii) $x=c^2$, $y=c^3$; (iii) $x=\frac{31}{4}c^2$, $y=\frac{31}{4}c^3$;

$(\phi=0)$ (i) $x=3p^2$, $y=2p^3$; (ii) $p=0$; (iii) $x=\frac{31}{9}p^2$, $y=\frac{62}{27}p^3$.

They are (i) a special curve of the family $c=$ constant; (ii) the cuspidal locus; (iii) the envelope.

The same argument extends to partial differential equations.